STRUCTRAL STABILITY AND DESIGN

Chapter 1. Introduction to Structural Stability

OUTLINE

- Definition of stability
- Types of instability
- Methods of stability analyses
- Examples – small deflection analyses
- Examples – large deflection analyses
- Examples – imperfect systems
- Design of steel structures

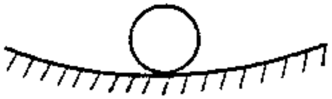
STABILITY DEFINITION

- Change in geometry of a structure or structural component under compression – resulting in loss of ability to resist loading is defined as *instability* in the book.
- Instability can lead to catastrophic failure → must be accounted in design. Instability is a strength-related limit state.
- Why did we define instability instead of stability? Seem strange!
- Stability is not easy to define.
  - Every structure is in equilibrium – static or dynamic. If it is not in equilibrium, the body will be in motion or a *mechanism*.
  - A *mechanism* cannot resist loads and is of no use to the civil engineer.
  - Stability qualifies the state of equilibrium of a structure. Whether it is in *stable* or *unstable* equilibrium.

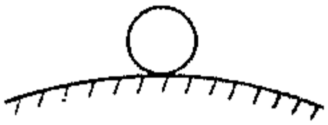
STABILITY DEFINITION

- Structure is in stable equilibrium when small perturbations do not cause large movements like a mechanism. Structure vibrates about it equilibrium position.
- Structure is in unstable equilibrium when small perturbations produce large movements – and the structure never returns to its original equilibrium position.
- Structure is in neutral equilibrium when we cant decide whether it is in stable or unstable equilibrium. Small perturbation cause large movements – but the structure can be brought back to its original equilibrium position with no work.
- Thus, stability talks about the equilibrium state of the structure.
- The definition of stability had nothing to do with a change in the geometry of the structure under compression – seems strange!

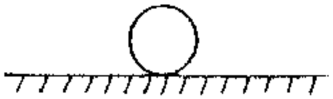
STABILITY DEFINITION



(a) STABLE EQUILIBRIUM



(b) UNSTABLE EQUILIBRIUM

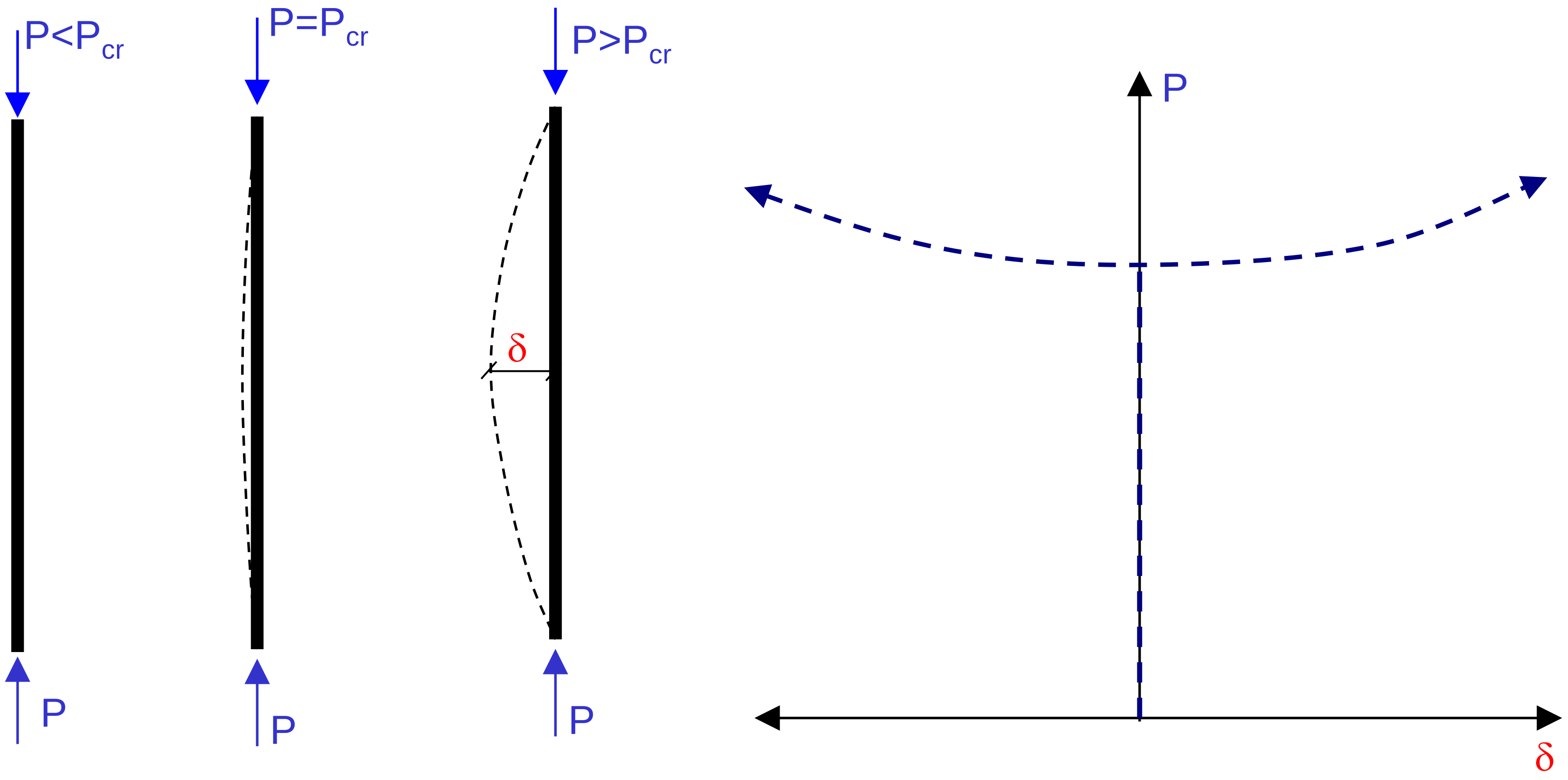


(c) NEUTRAL EQUILIBRIUM

BUCKLING Vs. STABILITY

- Change in geometry of structure under compression – that results in its ability to resist loads – called *instability*.
- Not true – this is called *buckling.*
- *Buckling* is a phenomenon that can occur for structures under compressive loads.
  - The structure deforms and is in stable equilibrium in state-1.
  - As the load increases, the structure suddenly changes to deformation state-2 at some critical load P_{cr}.
  - The structure buckles from state-1 to state-2, where state-2 is orthogonal (has nothing to do, or independent) with state-1.
- What has buckling to do with stability?
  - The question is - Is the equilibrium in state-2 stable or unstable?
  - Usually, state-2 after buckling is either neutral or unstable equilibrium

BUCKLING



BUCKLING Vs. STABILITY

- Thus, there are two topics we will be interested in this course
  - Buckling – Sudden change in deformation from state-1 to state-2
  - Stability of equilibrium – As the loads acting on the structure are increased, when does the equilibrium state become unstable?
  - The equilibrium state becomes unstable due to:
    - Large deformations of the structure
    - Inelasticity of the structural materials
- We will look at both of these topics for
  - Columns
  - Beams
  - Beam-Columns
  - Structural Frames

TYPES OF INSTABILITY

Structure subjected to compressive forces can undergo:

1. Buckling – bifurcation of equilibrium from deformation state-1 to state-2.
   - Bifurcation buckling occurs for columns, beams, and symmetric frames under gravity loads only
2. Failure due to instability of equilibrium state-1 due to large deformations or material inelasticity
   - Elastic instability occurs for beam-columns, and frames subjected to gravity and lateral loads.
   - Inelastic instability can occur for all members and the frame.

- We will study all of this in this course because we don't want our designed structure to buckle or fail by instability – both of which are <u>strength limit states</u>.

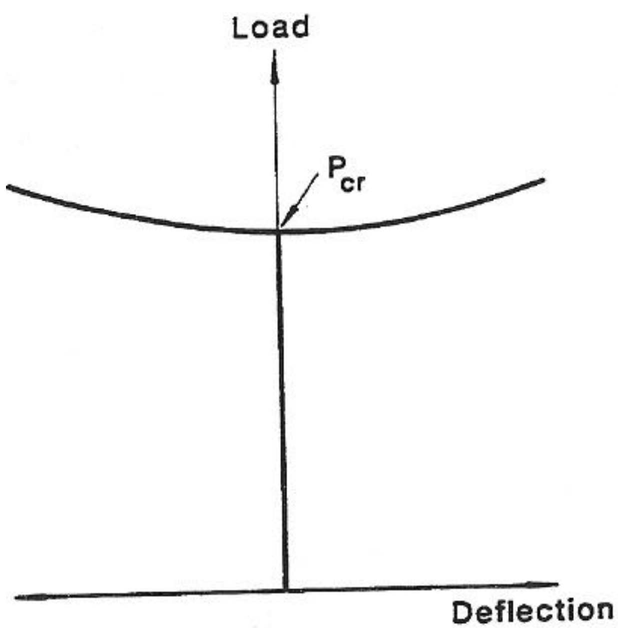
TYPES OF INSTABILITY

BIFURCATION BUCKLING

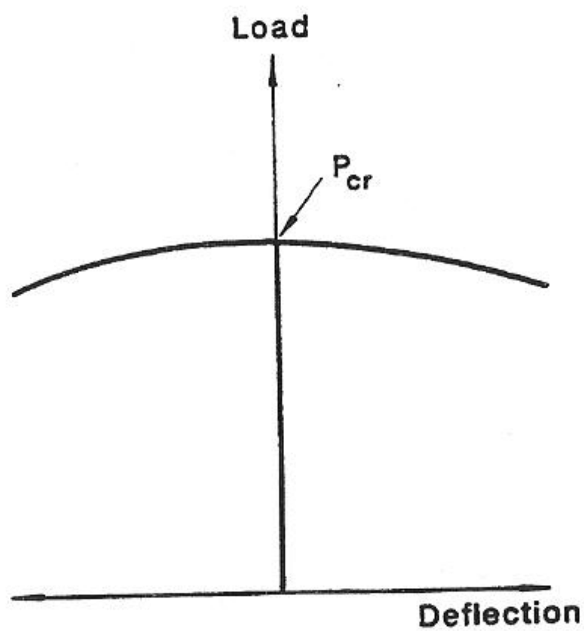
- Member or structure subjected to loads. As the load is increased, it reaches a *critical* value where:
  - The deformation changes suddenly from state-1 to state-2.
  - And, the equilibrium load-deformation path bifurcates.
- Critical buckling load when the load-deformation path bifurcates
  - Primary load-deformation path before buckling
  - Secondary load-deformation path post buckling
  - Is the post-buckling path stable or unstable?

SYMMETRIC BIFURCATION

- Post-buckling load-deform. paths are *symmetric* about load axis.
  - If the load capacity increases after buckling then stable symmetric bifurcation.
  - If the load capacity decreases after buckling then unstable symmetric bifurcation.



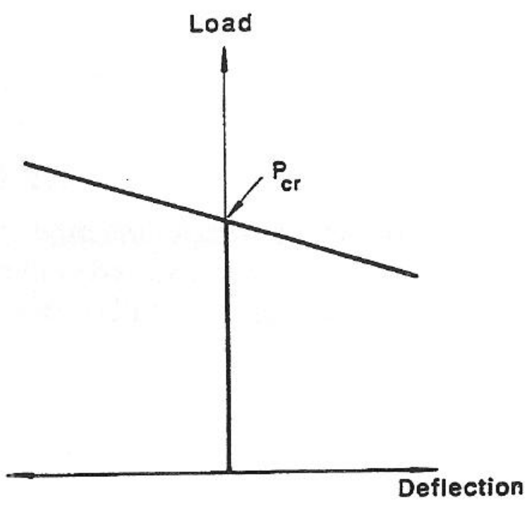
(a) STABLE SYMMETRIC BIFURCATION



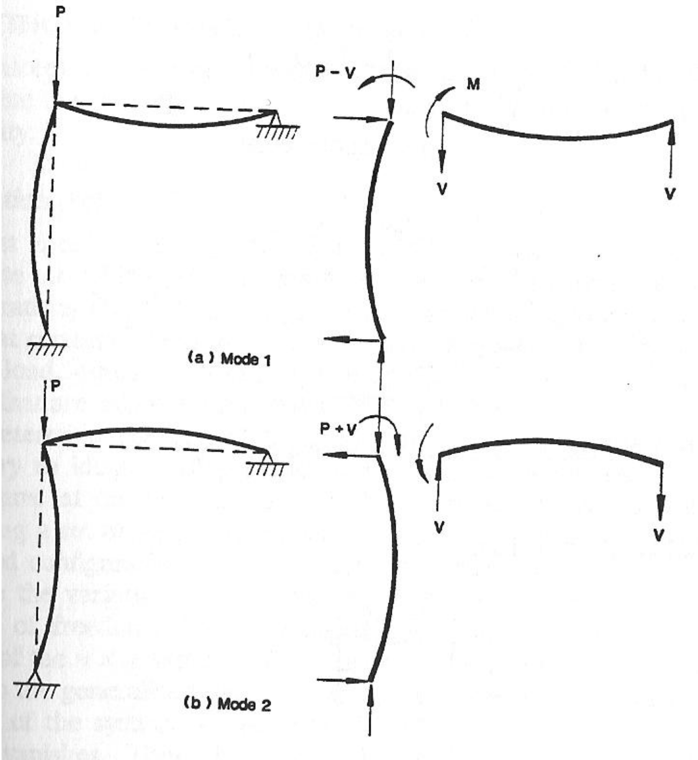
(b) UNSTABLE SYMMETRIC BIFURCATION

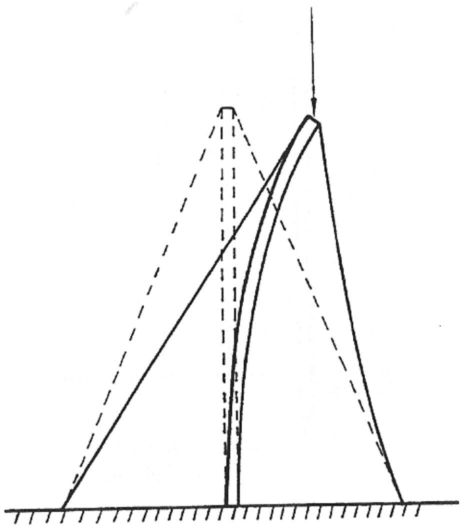
ASYMMETRIC BIFURCATION

- Post-buckling behavior that is asymmetric about load axis.



(c) ASYMMETRIC BIFURCATION





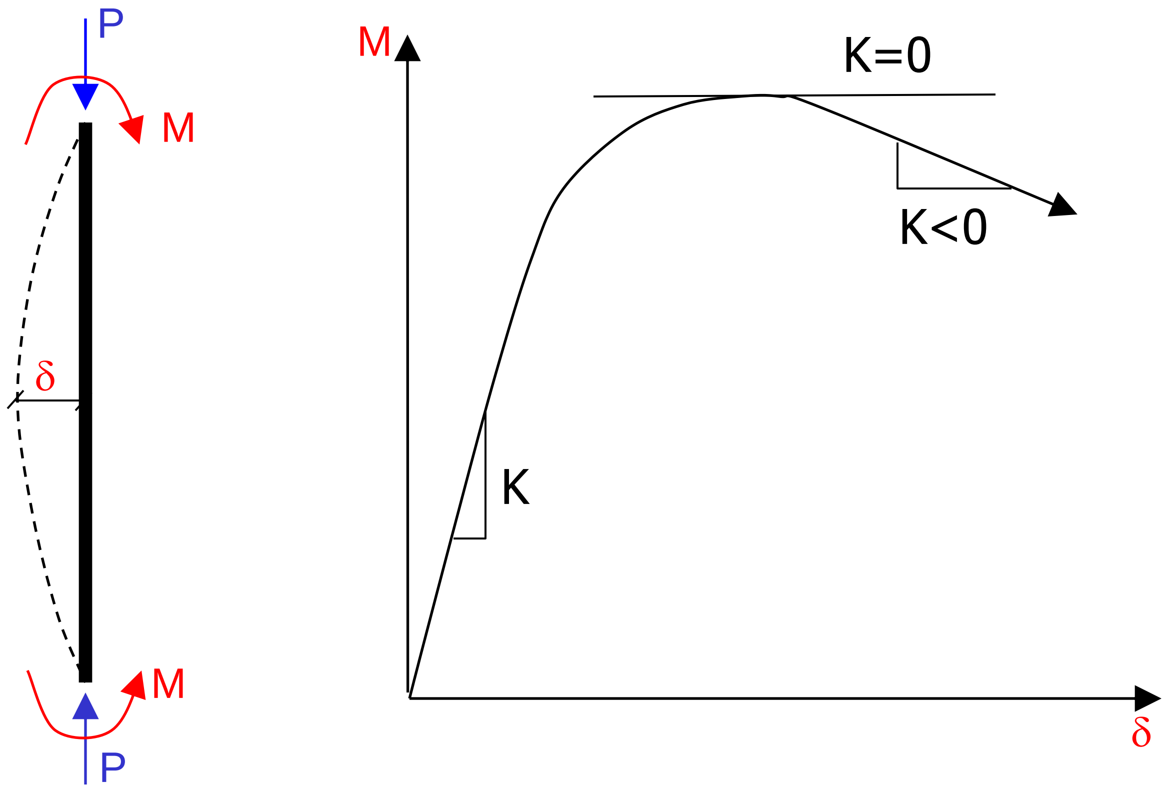
GUYED TOWER

INSTABILITY FAILURE

- There is no bifurcation of the load-deformation path. The deformation stays in state-1 throughout
- The structure stiffness decreases as the loads are increased. The change is stiffness is due to large deformations and / or material inelasticity.
  - The structure stiffness decreases to zero and becomes negative.
  - The load capacity is reached when the stiffness becomes zero.
  - Neutral equilibrium when stiffness becomes zero and unstable equilibrium when stiffness is negative.
  - Structural stability failure – when stiffness becomes negative.

INSTABILITY FAILURE

- FAILURE OF BEAM-COLUMNS

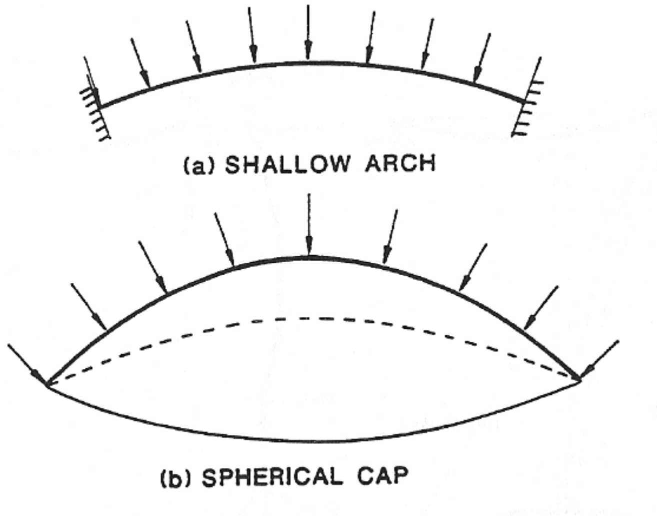


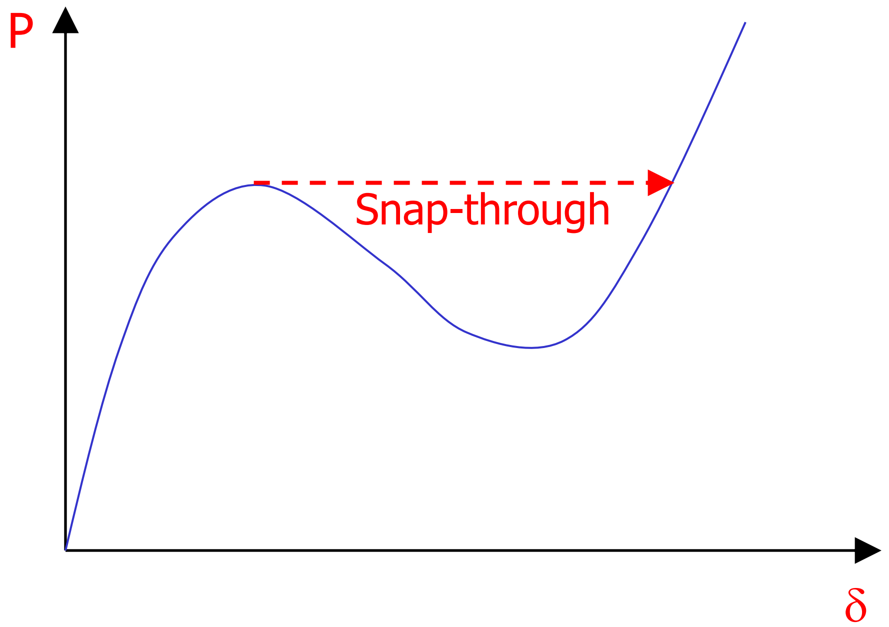
No bifurcation.

Instability due to <u>material and geometric nonlinearity</u>

INSTABILITY FAILURE

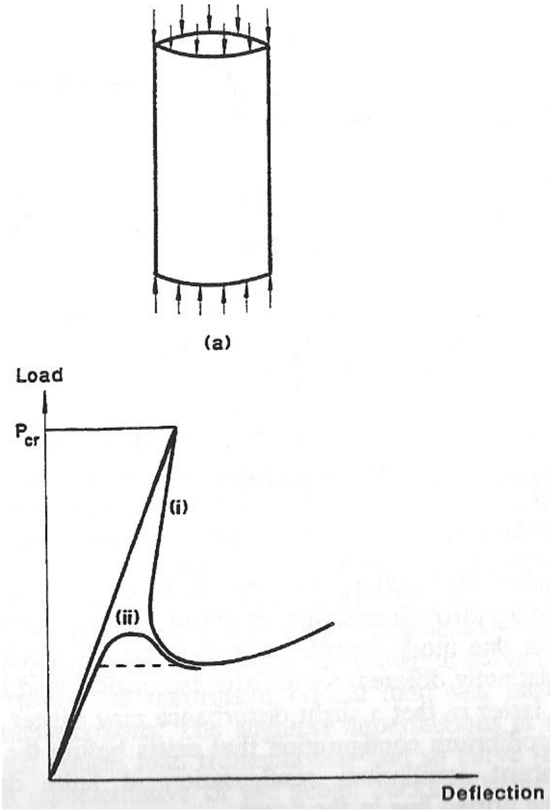
- Snap-through buckling





INSTABILITY FAILURE

- Shell Buckling failure – very sensitive to imperfections



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METHODS OF STABILITY ANALYSES

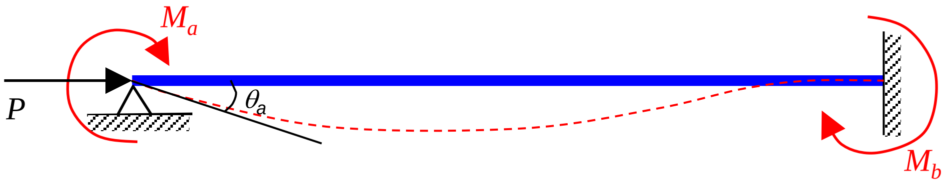
- Bifurcation approach – consists of writing the equation of equilibrium and solving it to determine the onset of buckling.
- Energy approach – consists of writing the equation expressing the complete potential energy of the system. Analyzing this total potential energy to establish equilibrium and examine stability of the equilibrium state.
- Dynamic approach – consists of writing the equation of dynamic equilibrium of the system. Solving the equation to determine the natural frequency (ω) of the system. Instability corresponds to the reduction of ω to zero.

STABILITY ANALYSES

- Each method has its advantages and disadvantages. In fact, you can use different methods to answer different questions
- The bifurcation approach is appropriate for determining the critical buckling load for a (perfect) system subjected to loads.
  - The deformations are usually assumed to be small.
  - The system must not have any imperfections.
  - It cannot provide any information regarding the post-buckling load-deformation path.
- The energy approach is the best when establishing the equilibrium equation and examining its stability
  - The deformations can be small or large.
  - The system can have imperfections.
  - It provides information regarding the post-buckling path if large deformations are assumed
  - The major limitation is that it requires the <u>assumption of the deformation state</u>, and it should include all possible degrees of freedom.

STABILITY ANALYSIS

- The dynamic method is very powerful, but we will not use it in this class at all.
  - Remember, it though when you take the course in dynamics or earthquake engineering
  - In this class, you will learn that the loads acting on a structure change its stiffness. This is significant – you have not seen it before.



$$M_a = \frac{4EI}{L}\theta_a \qquad M_b = \frac{2EI}{L}\theta_b$$

- What happens when an axial load is acting on the beam.
  - The stiffness will no longer remain 4EI/L and 2EI/L.
  - Instead, it will decrease. The reduced stiffness will reduce the natural frequency and period elongation.
  - You will see these in your dynamics and earthquake engineering class.

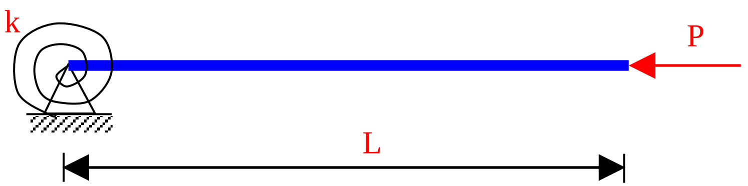
STABILITY ANALYSIS

- **FOR ANY KIND OF BUCKLING OR STABILITY ANALYSIS – NEED TO DRAW THE FREE BODY DIAGRAM OF THE DEFORMED STRUCTURE.**
- **WRITE THE EQUATION OF STATIC EQUILIBRIUM IN THE DEFORMED STATE**
- **WRITE THE ENERGY EQUATION IN THE DEFORMED STATE TOO.**
- **THIS IS CENTRAL TO THE TOPIC OF STABILITY ANALYSIS**
- **NO STABILITY ANALYSIS CAN BE PERFORMED IF THE FREE BODY DIAGRAM IS IN THE UNDEFORMED STATE**

BIFURCATION ANALYSIS

- Always a small deflection analysis
- To determine P_{cr} buckling load
- Need to assume buckled shape (state 2) to calculate

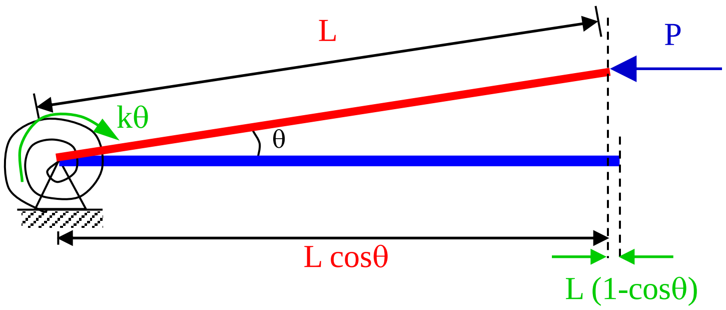
Example 1 – Rigid bar supported by rotational spring



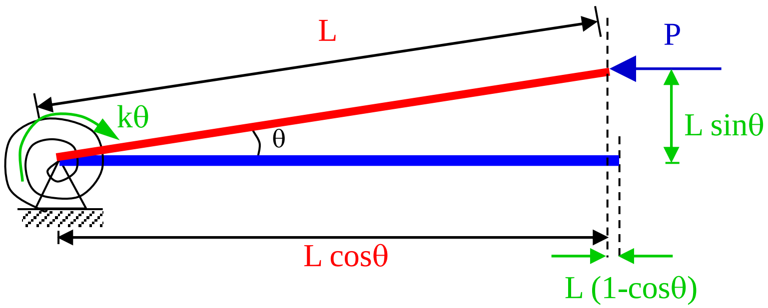
Rigid bar subjected to axial force P

Rotationally restrained at end

Step 1 - Assume a deformed shape that activates all possible d.o.f.



BIFURCATION ANALYSIS



- Write the equation of static equilibrium in the deformed state

$$\circlearrowleft + \sum M_o = 0 \qquad \therefore -k\theta + PL\sin\theta = 0$$

$$\therefore P = \frac{k\theta}{L\sin\theta}$$

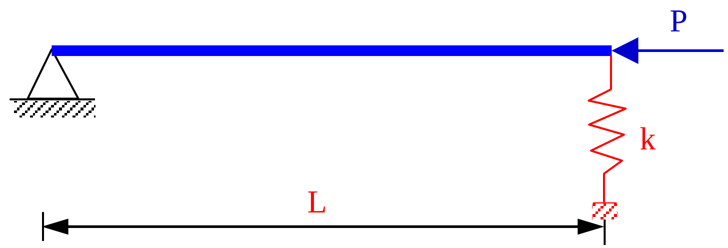
$$\textit{For small deformations} \quad \sin\theta = \theta$$

$$\therefore P_{cr} = \frac{k\theta}{L\theta} = \frac{k}{L}$$

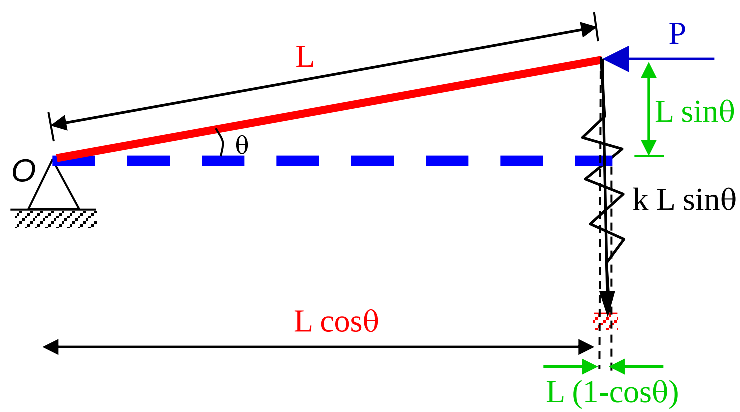
- Thus, the structure will be in static equilibrium in the deformed state when $P = P_{cr} = k/L$
- When $P<P_{cr}$, the structure will <u>not be</u> in the deformed state. The structure will buckle into the deformed state when $P=P_{cr}$

BIFURCATION ANALYSIS

Example 2 - Rigid bar supported by translational spring at end

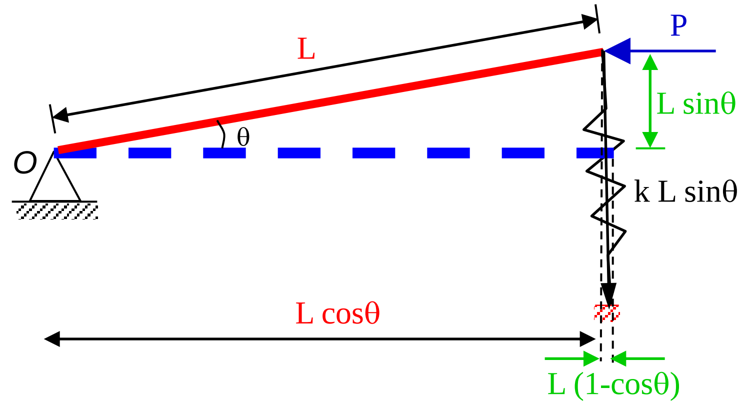


Assume deformed state that activates all possible d.o.f.
Draw FBD in the deformed state



BIFURCATION ANALYSIS

Write equations of static equilibrium in deformed state



$$+\sum M_o = 0 \qquad \therefore -(k\,L\sin\theta)\times L + P\,L\sin\theta = 0$$

$$\therefore P = \frac{k\,L^2 \sin\theta}{L\sin\theta}$$

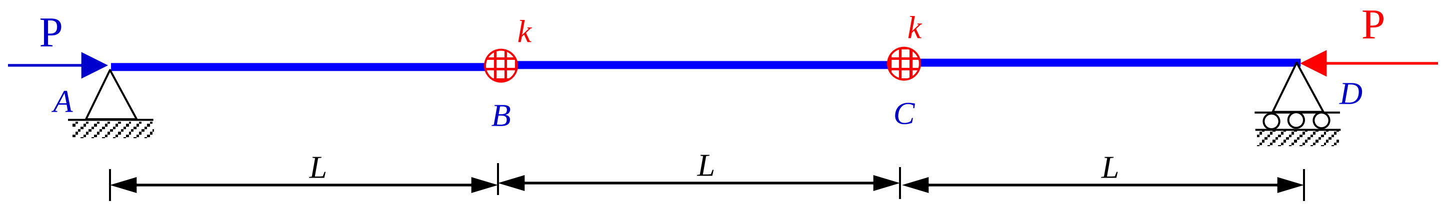
$$For\ small\ deformations \quad \sin\theta = \theta$$

$$\therefore P_{cr} = \frac{k\,L^2\theta}{L\theta} = k\,L$$

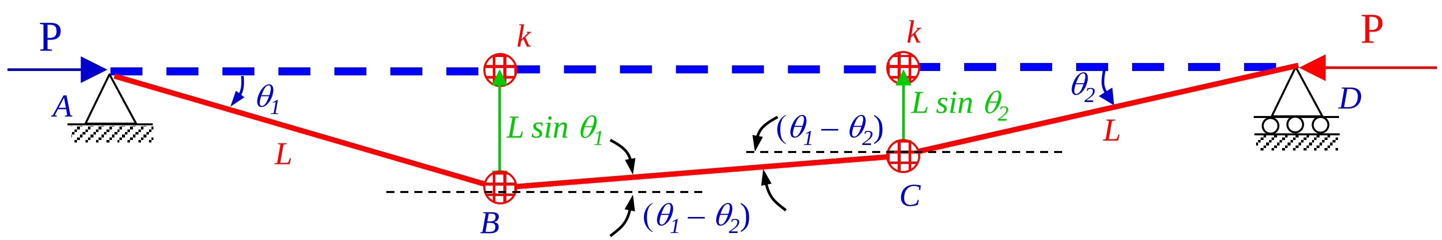
- Thus, the structure will be in static equilibrium in the deformed state when P = P_{cr} = k L. When P<Pcr, the structure will not be in the deformed state. The structure will buckle into the deformed state when P=P_{cr}

BIFURCATION ANALYSIS

Example 3 – Three rigid bar system with two rotational springs



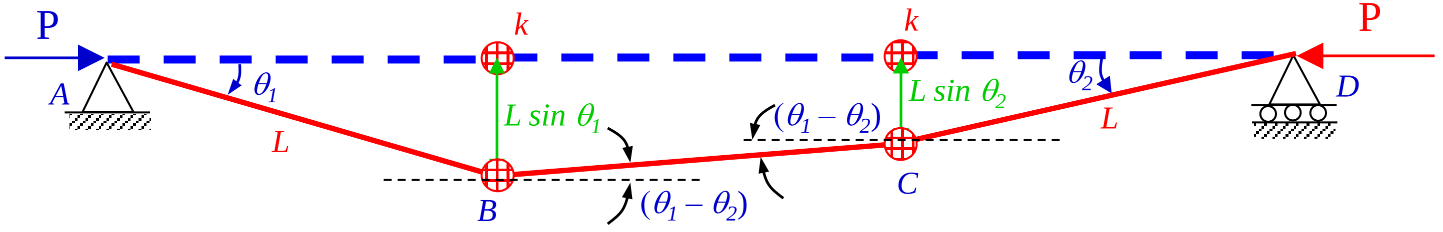
Assume deformed state that activates all possible d.o.f.
Draw FBD in the deformed state

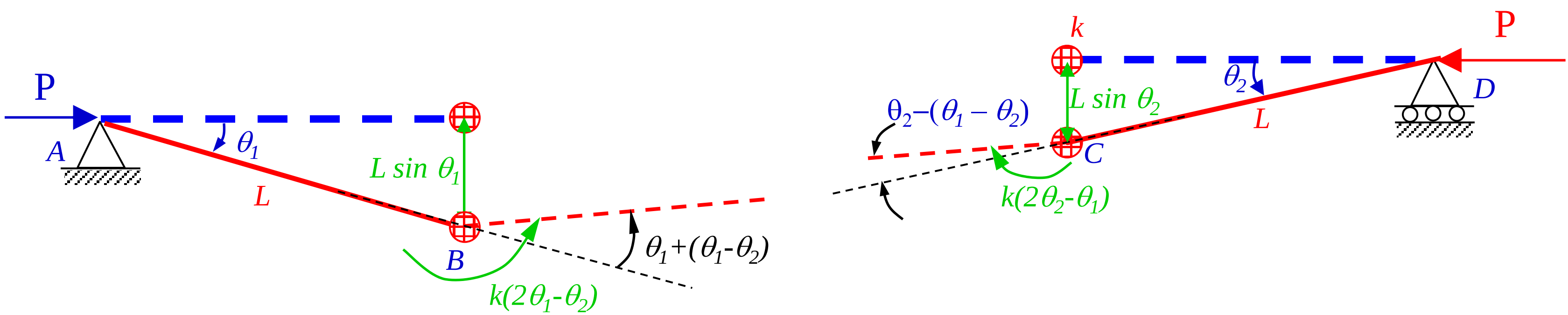


Assume small deformations. Therefore, sinθ=θ

BIFURCATION ANALYSIS

Write equations of static equilibrium in deformed state





$$\curvearrowleft + \sum M_B = 0 \qquad \therefore k(2\theta_1 - \theta_2) - PL\sin\theta_1 = 0 \qquad \therefore k(2\theta_1 - \theta_2) - PL\,\theta_1 = 0$$

$$\curvearrowleft + \sum M_C = 0 \qquad \therefore -k(2\theta_2 - \theta_1) + PL\sin\theta_2 = 0 \qquad \therefore -k(2\theta_2 - \theta_1) + PL\,\theta_2 = 0$$

BIFURCATION ANALYSIS

- Equations of Static Equilibrium

$$k(2\theta_1-\theta_2)-PL\,\theta_1=0$$

$$-k(2\theta_2-\theta_1)+PL\,\theta_2=0$$

$$\therefore \begin{bmatrix} 2k-PL & -k \\ -k & 2k-PL \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- Therefore either θ_1 and θ_2 are equal to zero or the determinant of the coefficient matrix is equal to zero.
- When θ_1 and θ_2 are not equal to zero – that is when buckling occurs – the coefficient matrix determinant has to be equal to zero for equil.
- Take a look at the matrix equation. It is of the form [A] {x}={0}. It can also be rewritten as ([K]-λ[I]){x}={0}

$$\therefore \left(\begin{bmatrix} \frac{2k}{L} & -\frac{k}{L} \\ -\frac{k}{L} & \frac{2k}{L} \end{bmatrix} - P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

BIFURCATION ANALYSIS

- This is the classical eigenvalue problem. $([K]-\lambda[I])\{x\}=\{0\}$.
- We are searching for the eigenvalues (λ) of the stiffness matrix [K]. These eigenvalues cause the stiffness matrix to become singular
  - Singular stiffness matrix means that it has a zero value, which means that the determinant of the matrix is equal to zero.

$$\begin{vmatrix} 2k-PL & -k \\ -k & 2k-PL \end{vmatrix} = 0$$

$$\therefore (2k-PL)^2 - k^2 = 0$$

$$\therefore (2k-PL+k)\bullet(2k-PL-k) = 0$$

$$\therefore (3k-PL)\bullet(k-PL) = 0$$

$$\therefore P_{cr} = \frac{3k}{L} \; or \; \frac{k}{L}$$

  - Smallest value of P_{cr} will govern. Therefore, $P_{cr}=k/L$

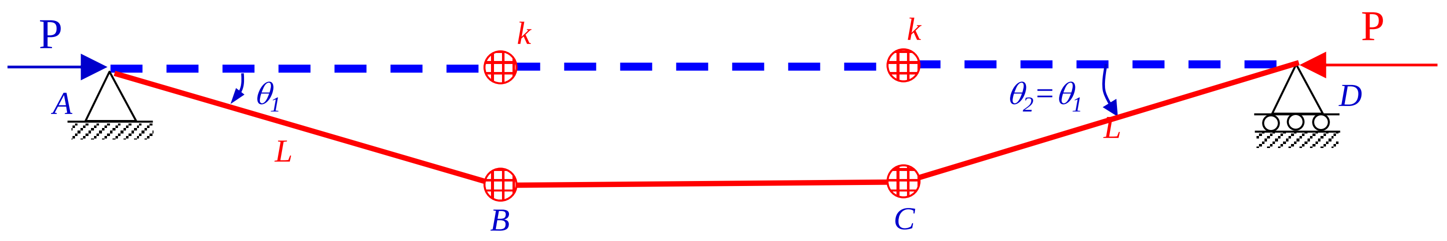
BIFURCATION ANALYSIS

- Each eigenvalue or critical buckling load (P_{cr}) corresponds to a buckling shape that can be determined as follows
- P_{cr}=k/L. Therefore substitute in the equations to determine θ_1 and θ_2

$$k(2\theta_1 - \theta_2) - PL\,\theta_1 = 0$$
$$Let\ P = P_{cr} = k/L$$
$$\therefore k(2\theta_1 - \theta_2) - k\theta_1 = 0$$
$$\therefore k\theta_1 - k\theta_2 = 0$$
$$\therefore \theta_1 = \theta_2$$

$$-k(2\theta_2 - \theta_1) + PL\,\theta_2 = 0$$
$$Let\ P = P_{cr} = k/L$$
$$\therefore -k(2\theta_2 - \theta_1) + k\theta_2 = 0$$
$$\therefore k\theta_1 - k\theta_2 = 0$$
$$\therefore \theta_1 = \theta_2$$

- All we could find is the relationship between θ_1 and θ_2. Not their specific values. Remember that this is a small deflection analysis. So, the values are negligible. What we have found is the buckling shape – not its magnitude.
- The buckling mode is such that θ_1=θ_2 → Symmetric buckling mode



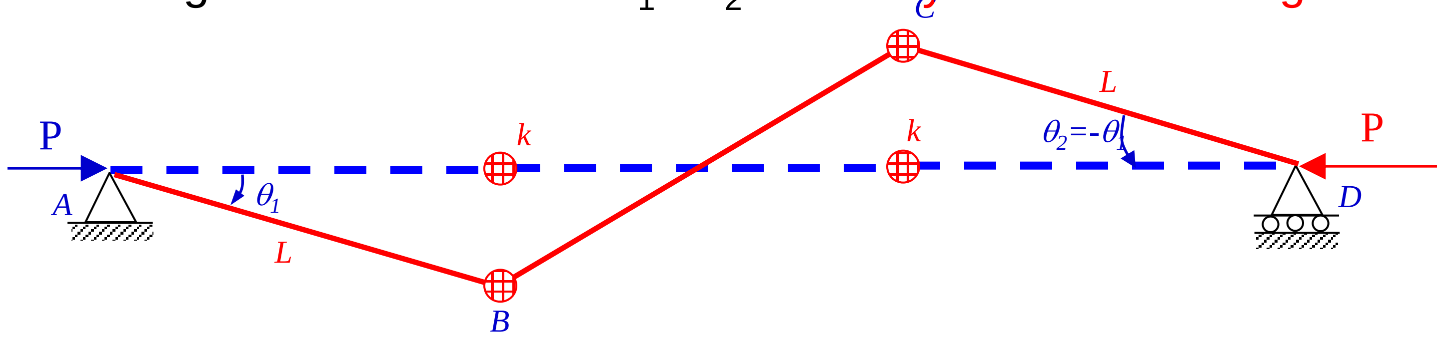
BIFURCATION ANALYSIS

- Second eigenvalue was P_{cr}=3k/L. Therefore substitute in the equations to determine θ_1 and θ_2

$$k(2\theta_1-\theta_2)-PL\,\theta_1=0$$
$$Let\ P=P_{cr}=3k/L$$
$$\therefore k(2\theta_1-\theta_2)-3k\theta_1=0$$
$$\therefore -k\theta_1-k\theta_2=0$$
$$\therefore \theta_1=-\theta_2$$

$$-k(2\theta_2-\theta_1)+PL\,\theta_2=0$$
$$Let\ P=P_{cr}=3k/L$$
$$\therefore -k(2\theta_2-\theta_1)+3k\theta_2=0$$
$$\therefore k\theta_1+k\theta_2=0$$
$$\therefore \theta_1=-\theta_2$$

- All we could find is the relationship between θ_1 and θ_2. Not their specific values. Remember that this is a small deflection analysis. So, the values are negligible. What we have found is the buckling shape – not its magnitude.
- The buckling mode is such that $\theta_1=-\theta_2$ → Antisymmetric buckling mode



BIFURCATION ANALYSIS

- Homework No. 1
  - Problem 1.1
  - Problem 1.3
  - Problem 1.4
  - All problems from the textbook on Stability by W.F. Chen

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- Bifurcation analysis examples – small deflection analyses
- Energy method
  - Examples – small deflection analyses
  - Examples – large deflection analyses
  - Examples – imperfect systems
- Design of steel structures

ENERGY METHOD

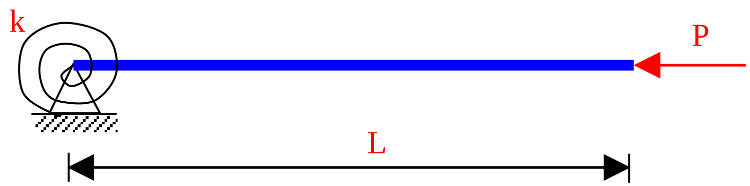
- We will currently look at the use of the energy method for an elastic system subjected to conservative forces.
- Total potential energy of the system – Π – depends on the work done by the external forces (W_e) and the strain energy stored in the system (U).
- $\Pi = U - W_e$.
- For the system to be in equilibrium, its total potential energy Π must be stationary. That is, the first derivative of Π must be equal to zero.
- Investigate higher order derivatives of the total potential energy to examine the stability of the equilibrium state, i.e., whether the equilibrium is stable or unstable

ENERGY METHD

- The energy method is the best for establishing the equilibrium equation and examining its stability
  - The deformations can be small or large.
  - The system can have imperfections.
  - It provides information regarding the post-buckling path if large deformations are assumed
  - The major limitation is that it requires the <u>assumption of the deformation state</u>, and it should include all possible degrees of freedom.

ENERGY METHOD

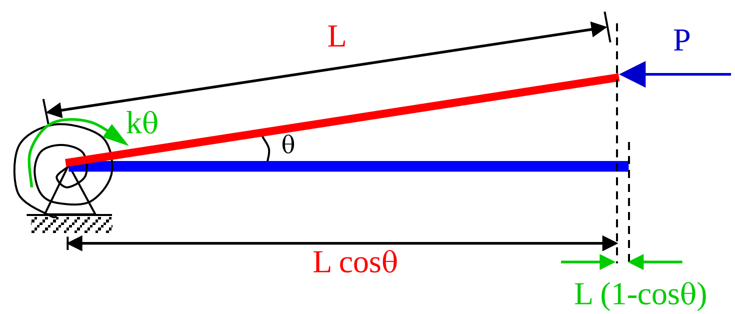
- Example 1 – Rigid bar supported by rotational spring
- Assume small deflection theory



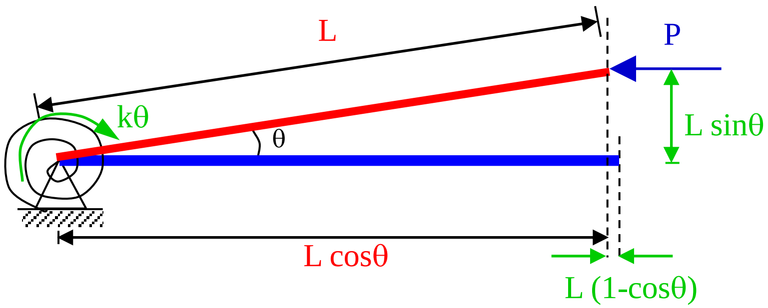
Rigid bar subjected to axial force P

Rotationally restrained at end

Step 1 - Assume a deformed shape that activates all possible d.o.f.



ENERGY METHOD – SMALL DEFLECTIONS



- Write the equation representing the total potential energy of system

$$\prod = U - W_e$$

$$U = \frac{1}{2} k\,\theta^2$$

$$W_e = P\,L\,(1 - \cos\theta)$$

$$\prod = \frac{1}{2} k\,\theta^2 - P\,L\,(1 - \cos\theta)$$

$$\frac{d\prod}{d\theta} = k\,\theta - P\,L \sin\theta$$

$$For\ equilibrium;\ \frac{d\prod}{d\theta} = 0$$

$$Therefore, \qquad k\,\theta - P\,L \sin\theta = 0$$

$$For\ small\ deflections;\ k\theta - P\,L\theta = 0$$

$$Therefore,\ P_{cr} = \frac{k}{L}$$

ENERGY METHOD – SMALL DEFLECTIONS

- The energy method predicts that buckling will occur at the same load P_{cr} as the bifurcation analysis method.
- At P_{cr}, the system will be in equilibrium in the deformed.
- Examine the stability by considering further derivatives of the total potential energy
  - This is a small deflection analysis. Hence θ will be → zero.
  - In this type of analysis, the further derivatives of Π examine the stability of the initial state-1 (when $\theta = 0$)

$$\Pi = \frac{1}{2} k\,\theta^2 - P\,L\,(1 - \cos\theta)$$

$$\frac{d\Pi}{d\theta} = k\,\theta - P\,L \sin\theta = k\,\theta - P\,L\,\theta$$

$$\frac{d^2\Pi}{d\theta^2} = k - PL$$

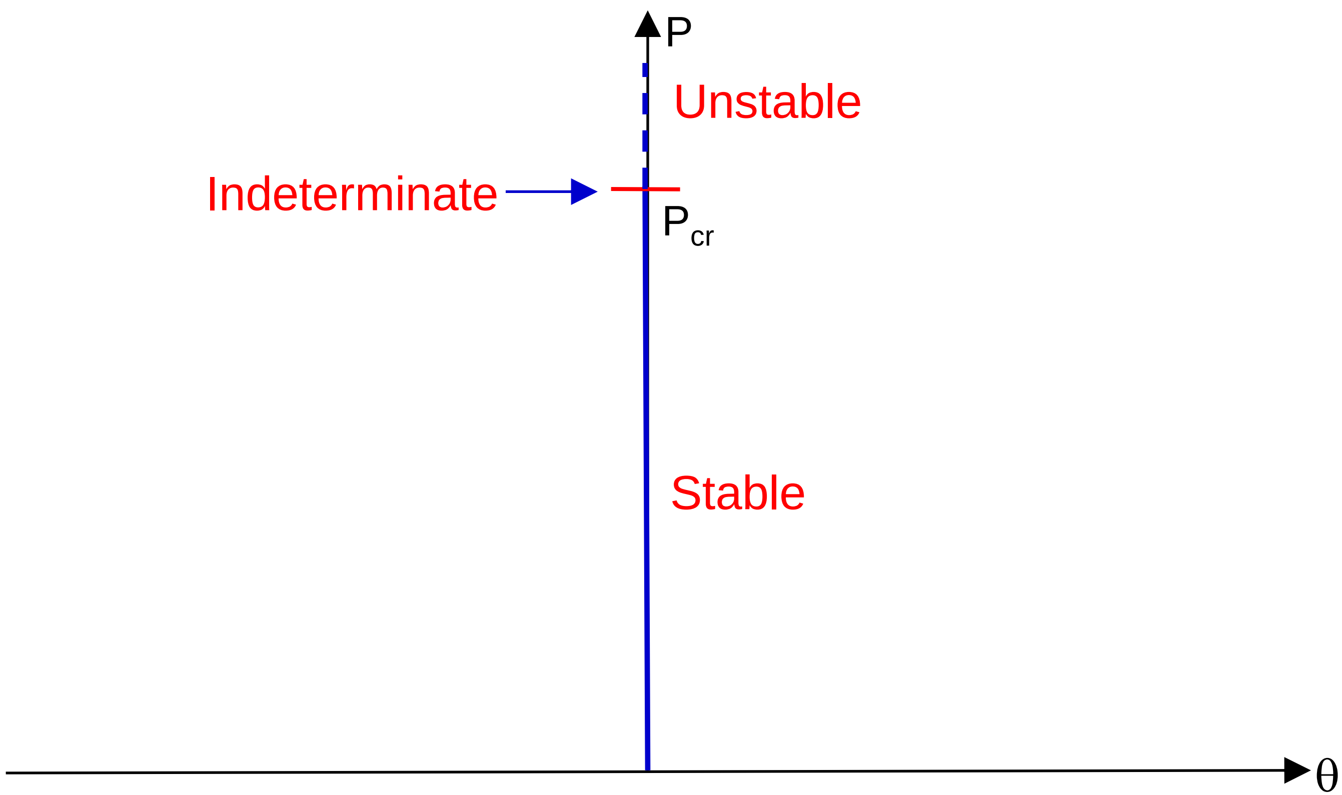
$$When\ P < P_{cr} \quad \frac{d^2\Pi}{d\theta^2} > 0 \quad \therefore Stable\ equilibrium$$

$$When\ P > P_{cr} \quad \frac{d^2\Pi}{d\theta^2} < 0 \quad \therefore Unstable\ equilibrium$$

$$When\ P = P_{cr} \quad \frac{d^2\Pi}{d\theta^2} = 0 \quad \therefore\ Not\ sure$$

ENERGY METHOD – SMALL DEFLECTIONS

- In state-1, stable when $P<P_{cr}$, unstable when $P>P_{cr}$
- No idea about state during buckling.
- No idea about post-buckling equilibrium path or its stability.



ENERGY METHOD – LARGE DEFLECTIONS

- Example 1 – Large deflection analysis (rigid bar with rotational spring)

$$\prod = U - W_e$$

$$U = \frac{1}{2} k\,\theta^2$$

$$W_e = P\,L\,(1-\cos\theta)$$

$$\prod = \frac{1}{2} k\,\theta^2 - P\,L\,(1-\cos\theta)$$

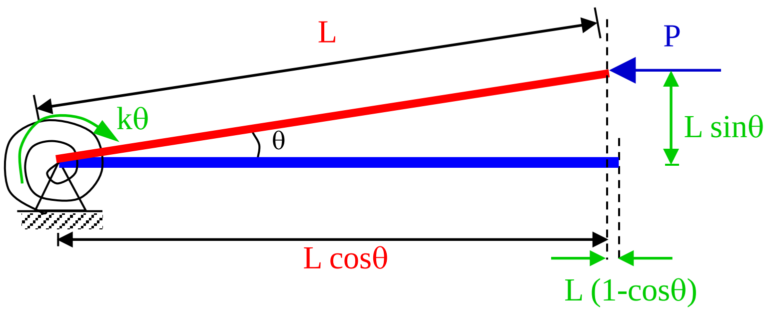
$$\frac{d\prod}{d\theta} = k\,\theta - P\,L\sin\theta$$

$$For\ equilibrium;\ \frac{d\prod}{d\theta} = 0$$

$$Therefore,\qquad k\,\theta - P\,L\sin\theta = 0$$

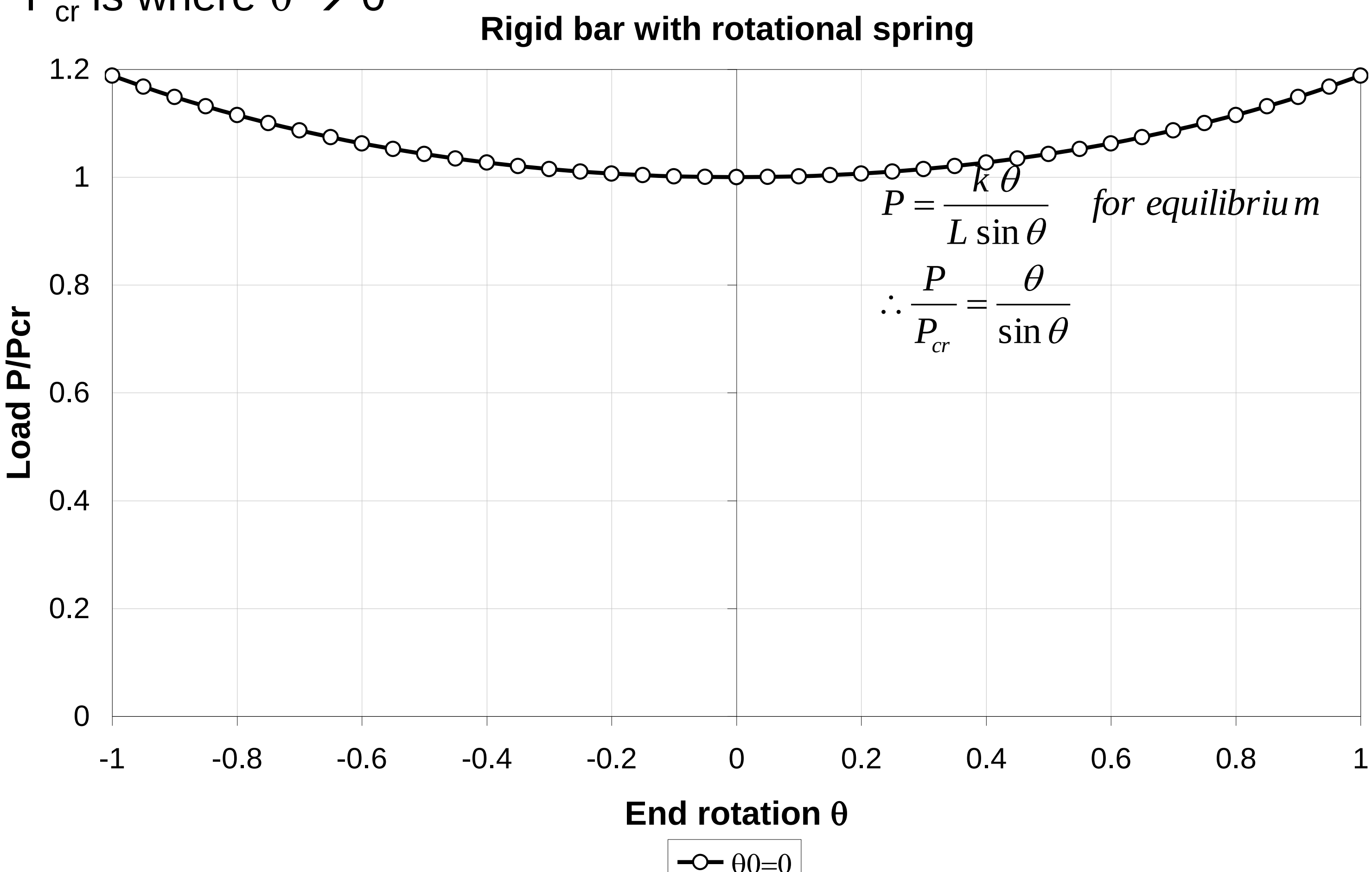
$$Therefore,\qquad P = \frac{k\,\theta}{L\sin\theta}\quad for\ equilibrium$$

$$The\ post-buckling\ P-\theta\ relationship\ is\ given\ above$$



ENERGY METHOD – LARGE DEFLECTIONS

- Large deflection analysis
  - See the post-buckling load-displacement path shown below
  - The load carrying capacity increases after buckling at P_{cr}
  - P_{cr} is where $\theta \rightarrow 0$



$$P = \frac{k\,\theta}{L\sin\theta} \quad \textit{for equilibrium}$$

$$\therefore \frac{P}{P_{cr}} = \frac{\theta}{\sin\theta}$$

ENERGY METHOD – LARGE DEFLECTIONS

- Large deflection analysis – Examine the stability of equilibrium using higher order derivatives of Π

$$\Pi = \frac{1}{2} k\, \theta^2 - P\, L(1 - \cos\theta)$$

$$\frac{d\Pi}{d\theta} = k\, \theta - P\, L \sin\theta$$

$$\frac{d^2\Pi}{d\theta^2} = k - P\, L \cos\theta$$

$$But,\ P = \frac{k\, \theta}{L \sin\theta}$$

$$\therefore \frac{d^2\Pi}{d\theta^2} = k - \frac{k\theta}{L \sin\theta} L \cos\theta$$

$$\therefore \frac{d^2\Pi}{d\theta^2} = k\left(1 - \frac{\theta}{\tan\theta}\right)$$

$$\therefore \frac{d^2\Pi}{d\theta^2} > 0 \quad Always\ (i.e., all\ values\ of\ \theta)$$

$$\therefore \underline{Always\ STABLE}$$

$$But,\ \frac{d^2\Pi}{d\theta^2} = 0\ \ for\ \ \theta = 0$$

ENERGY METHOD – LARGE DEFLECTIONS

- At θ =0, the second derivative of Π=0. Therefore, inconclusive.
- Consider the Taylor series expansion of Π at θ=0

$$\Pi = \Pi\Big|_{\theta=0} + \frac{d\Pi}{d\theta}\Big|_{\theta=0}\theta + \frac{1}{2!}\frac{d^2\Pi}{d\theta^2}\Big|_{\theta=0}\theta^2 + \frac{1}{3!}\frac{d^3\Pi}{d\theta^3}\Big|_{\theta=0}\theta^3 + \frac{1}{4!}\frac{d^4\Pi}{d\theta^4}\Big|_{\theta=0}\theta^4 + ... + \frac{1}{n!}\frac{d^n\Pi}{d\theta^n}\Big|_{\theta=0}\theta^n$$

- Determine the first non-zero term of Π,

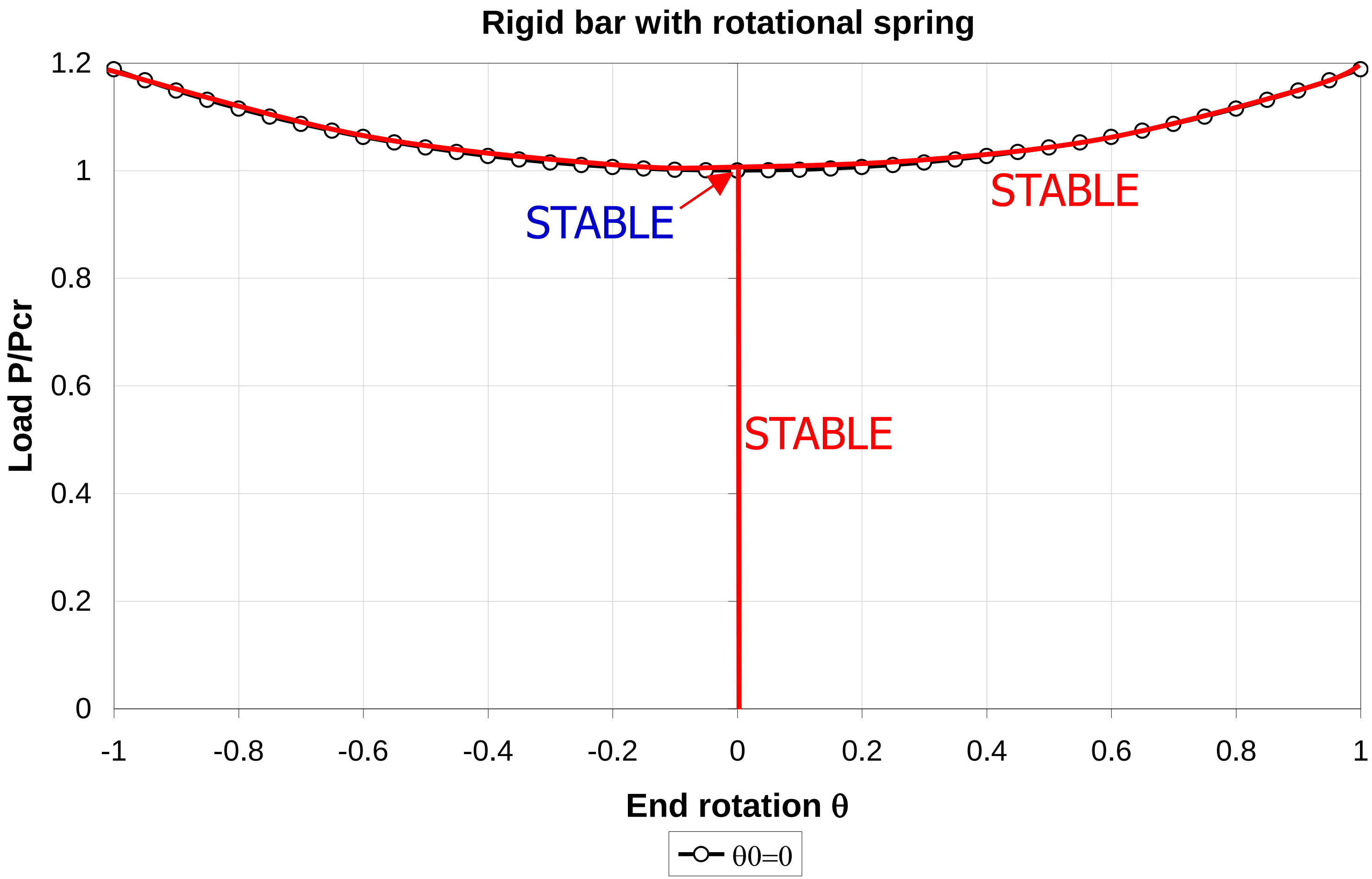
$$\Pi = \frac{1}{2}k\,\theta^2 - P\,L(1-\cos\theta)$$
$$\frac{d\Pi}{d\theta} = k\,\theta - P\,L\sin\theta$$
$$\frac{d^2\Pi}{d\theta^2} = k - P\,L\cos\theta$$
$$\frac{d^3\Pi}{d\theta^3} = P\,L\sin\theta$$
$$\frac{d^4\Pi}{d\theta^4} = P\,L\cos\theta$$

$$\Pi\Big|_{\theta=0} = 0$$
$$\frac{d\Pi}{d\theta}\Big|_{\theta=0} = 0$$
$$\frac{d^2\Pi}{d\theta^2}\Big|_{\theta=0} = 0$$
$$\frac{d^3\Pi}{d\theta^3}\Big|_{\theta=0} = P\,L\sin\theta = 0$$
$$\frac{d^4\Pi}{d\theta^4}\Big|_{\theta=0} = P\,L\cos\theta = PL = k$$

$$\therefore \frac{1}{4!}\frac{d^4\Pi}{d\theta^4}\Big|_{\theta=0}\theta^4 = \frac{1}{24}k\,\theta^4 > 0$$

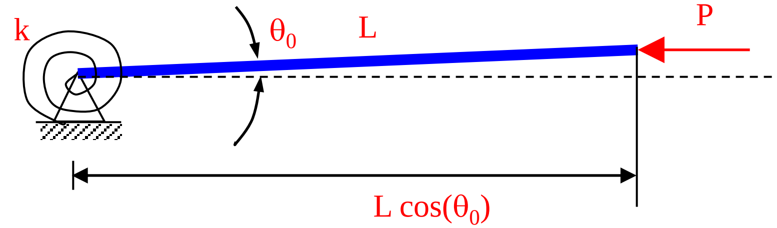
- Since the first non-zero term is > 0, the state is stable at $P=P_{cr}$ and θ=0

ENERGY METHOD – LARGE DEFLECTIONS

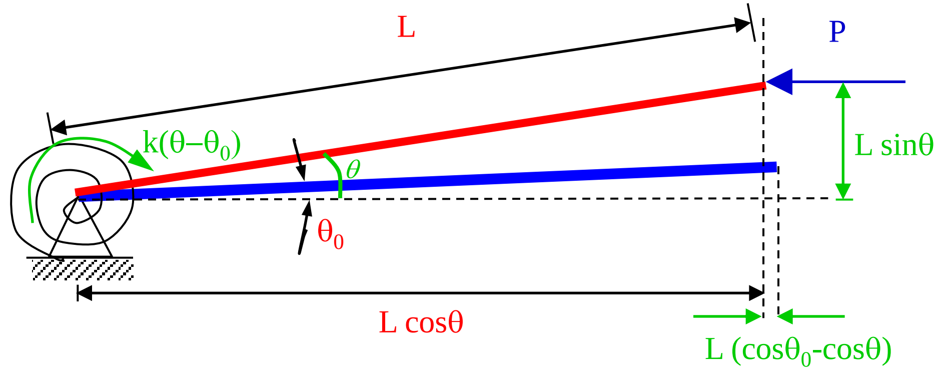


ENERGY METHOD – IMPERFECT SYSTEMS

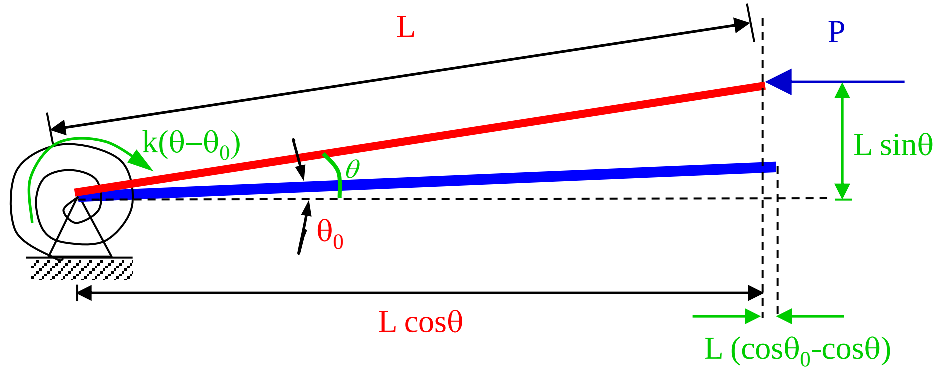
- Consider example 1 – but as a system with imperfections
  - The initial imperfection given by the angle θ_0 as shown below



  - The free body diagram of the deformed system is shown below



ENERGY METHOD – IMPERFECT SYSTEMS



$$\prod = U - W_e$$

$$U = \frac{1}{2} k\,(\theta - \theta_0)^2$$

$$W_e = P\,L\,(\cos\theta_0 - \cos\theta)$$

$$\prod = \frac{1}{2} k\,(\theta - \theta_0)^2 - P\,L\,(\cos\theta_0 - \cos\theta)$$

$$\frac{d\prod}{d\theta} = k\,(\theta - \theta_0) - P\,L\sin\theta$$

For equilibrium; $\frac{d\prod}{d\theta} = 0$

Therefore, $k\,(\theta - \theta_0) - P\,L\sin\theta = 0$

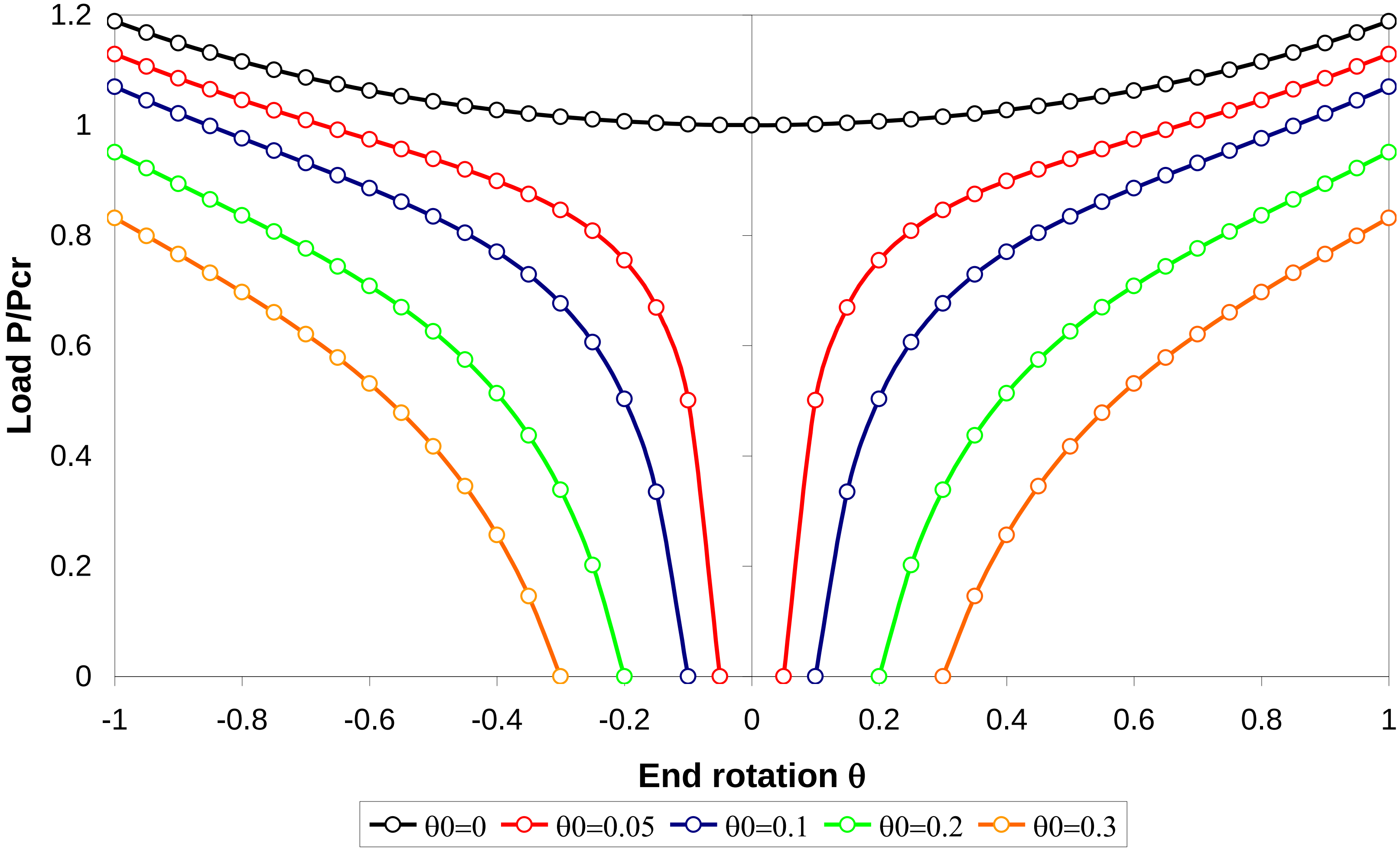
Therefore, $P = \frac{k\,(\theta - \theta_0)}{L\sin\theta}$ *for equilibrium*

The equilibrium $P - \theta$ *relationship is given above*

ENERGY METHOD – IMPERFECT SYSTEMS

$$P = \frac{k\,(\theta - \theta_0)}{L \sin\theta} \qquad \therefore \frac{P}{P_{cr}} = \frac{\theta - \theta_0}{\sin\theta}$$

$P - \theta$ *relationships for different values of* θ_0 *shown below:*



ENERGY METHODS – IMPERFECT SYSTEMS

- As shown in the figure, deflection starts as soon as loads are applied. There is no bifurcation of load-deformation path for imperfect systems. The load-deformation path remains in the same state through-out.
- The smaller the imperfection magnitude, the close the load-deformation paths to the perfect system load –deformation path
- The magnitude of load, is influenced significantly by the imperfection magnitude.
- All real systems have imperfections. They may be very small but will be there
- The magnitude of imperfection is not easy to know or guess. Hence if a perfect system analysis is done, the results will be close for an imperfect system with small imperfections

ENERGY METHODS – IMPERFECT SYSTEMS

- Examine the stability of the imperfect system using higher order derivatives of Π

$$\Pi = \frac{1}{2} k\,(\theta - \theta_0)^2 - P\,L\,(\cos\theta_0 - \cos\theta)$$

$$\frac{d\,\Pi}{d\theta} = k\,(\theta - \theta_0) - P\,L\sin\theta$$

$$\frac{d^2\,\Pi}{d\theta^2} = k - P\,L\cos\theta$$

$$\therefore \textit{Equilibrium path will be stable}$$

$$\textit{if}\ \ \frac{d^2\,\Pi}{d\theta^2} > 0$$

$$\textit{i.e., if}\quad k - P\,L\cos\theta > 0$$

$$\textit{i.e., if}\quad P < \frac{k}{L\cos\theta}$$

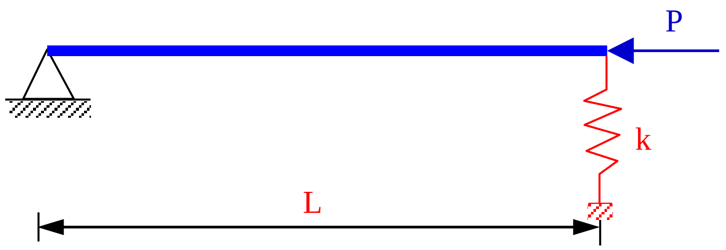
$$\textit{i.e., if}\ \frac{k(\theta - \theta_0)}{L\sin\theta} < \frac{k}{L\cos\theta}$$

$$\textit{i.e.,}\ \theta - \theta_0 < \tan\theta$$

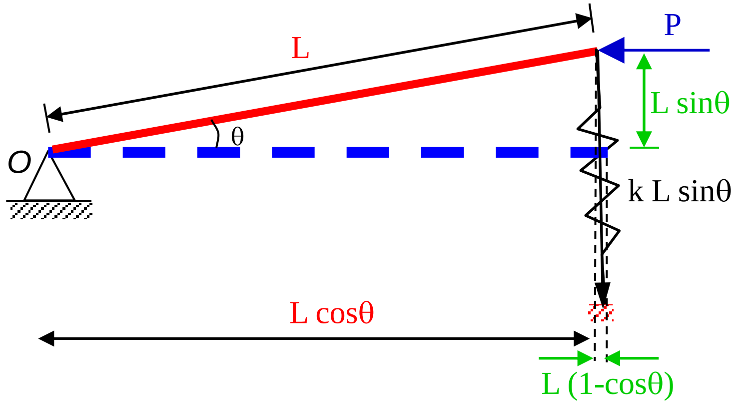
- Which is always true, hence always in STABLE EQUILIBRIUM

ENERGY METHOD – SMALL DEFLECTIONS

Example 2 - Rigid bar supported by translational spring at end



Assume deformed state that activates all possible d.o.f.
Draw FBD in the deformed state



ENERGY METHOD – SMALL DEFLECTIONS

Write the equation representing the total potential energy of system

$$\prod = U - W_e$$

$$U = \frac{1}{2} k \, (L \sin\theta)^2 = \frac{1}{2} k \, L^2 \theta^2$$

$$W_e = P \, L (1 - \cos\theta)$$

$$\prod = \frac{1}{2} k \, L^2 \, \theta^2 - P \, L (1 - \cos\theta)$$

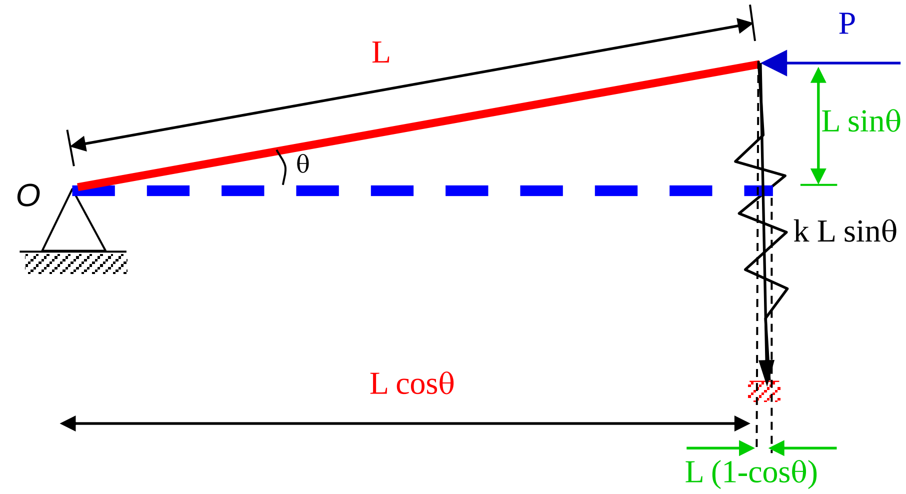
$$\frac{d\prod}{d\theta} = k \, L^2 \, \theta - P \, L \sin\theta$$

$$For\ equilibrium; \frac{d\prod}{d\theta} = 0$$

$$Therefore, \qquad k \, L^2 \, \theta - P \, L \sin\theta = 0$$

$$For\ small\ deflections; k \, L^2 \theta - P \, L \theta = 0$$

$$Therefore, P_{cr} = k \, L$$



ENERGY METHOD – SMALL DEFLECTIONS

- The energy method predicts that buckling will occur at the same load P_{cr} as the bifurcation analysis method.
- At P_{cr}, the system will be in equilibrium in the deformed. Examine the stability by considering further derivatives of the total potential energy
  - This is a small deflection analysis. Hence θ will be → zero.
  - In this type of analysis, the further derivatives of Π examine the stability of the initial state-1 (when θ =0)

$$\Pi = \frac{1}{2} k\ L^2\ \theta^2 - P\ L\,(1-\cos\theta)$$

$$\frac{d\Pi}{d\theta} = k\ L^2\ \theta - P\ L \sin\theta$$

$$\frac{d^2\Pi}{d\theta^2} = k\ L^2 - P\ L\ \cos\theta$$

$$\textit{For small deflections and } \theta = 0$$

$$\frac{d^2\Pi}{d\theta^2} = k\ L^2 - P\ L$$

$$\textit{When, } P < k\ L \qquad \frac{d^2\Pi}{d\theta^2} > 0 \quad \therefore \textit{STABLE}$$

$$\textit{When, } P > k\ L \qquad \frac{d^2\Pi}{d\theta^2} < 0 \quad \therefore \textit{UNSTABLE}$$

$$\textit{When } P = kL \qquad \frac{d^2\Pi}{d\theta^2} = 0 \quad \therefore \textit{INDETERMINATE}$$

ENERGY METHOD – LARGE DEFLECTIONS

Write the equation representing the total potential energy of system

$$\prod = U - W_e$$

$$U = \frac{1}{2} k\, (L \sin\theta)^2$$

$$W_e = P\, L\, (1 - \cos\theta)$$

$$\prod = \frac{1}{2} k\, L^2\, \sin^2\theta - P\, L\, (1 - \cos\theta)$$

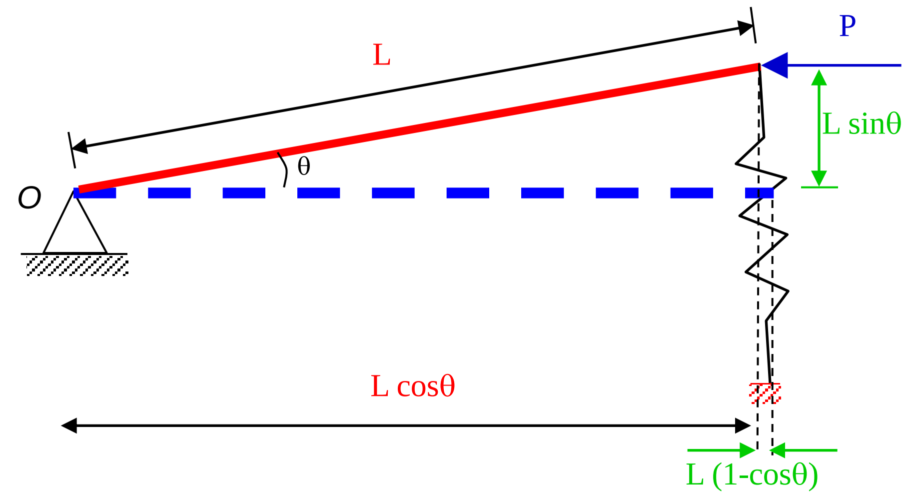
$$\frac{d\prod}{d\theta} = k\, L^2\, \sin\theta \cos\theta - P\, L \sin\theta$$

$$For\ equilibrium;\ \frac{d\prod}{d\theta} = 0$$

$$Therefore, \qquad k\, L^2\, \sin\theta\cos\theta - P\, L \sin\theta = 0$$

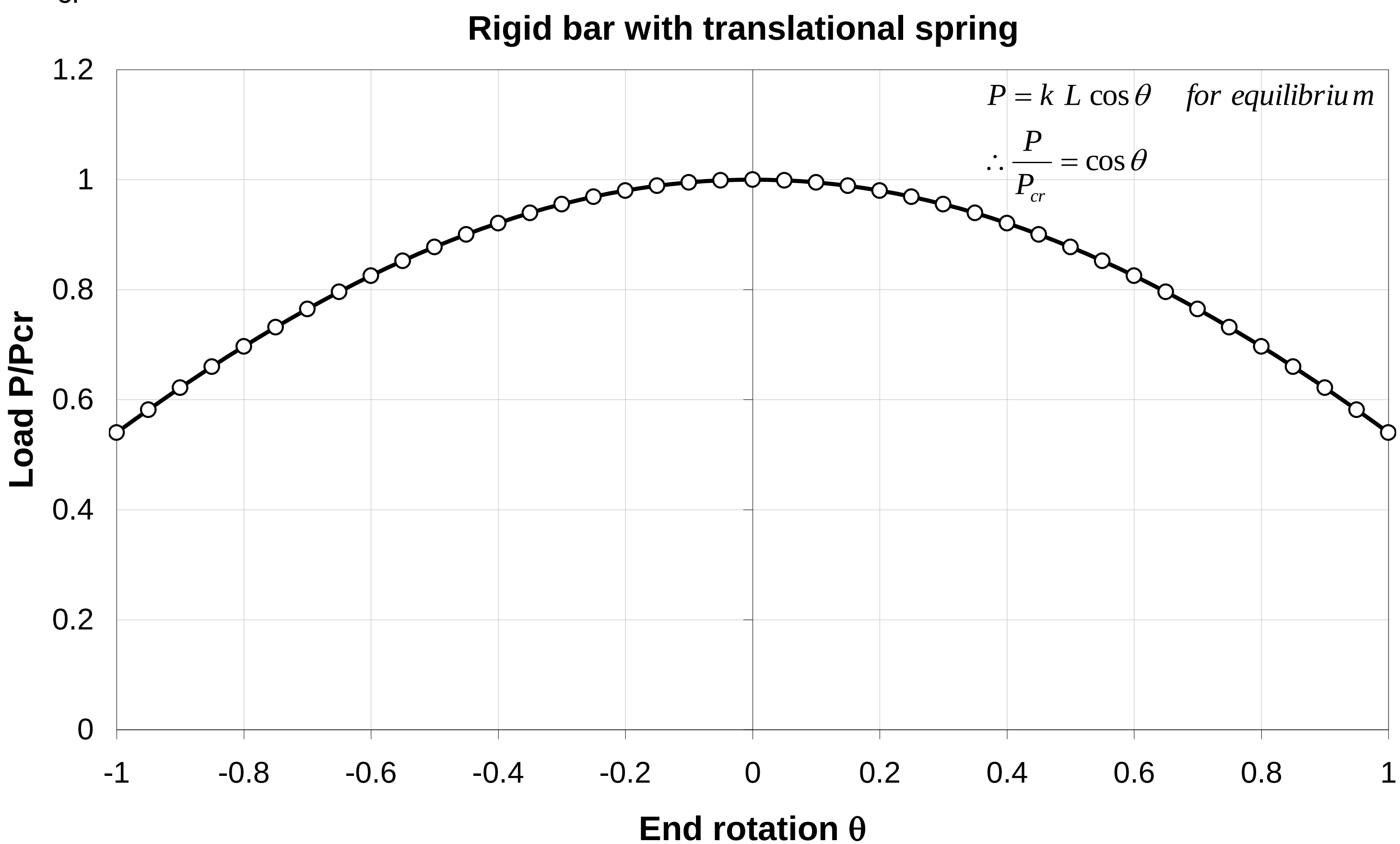
$$Therefore, \qquad P = k\, L\cos\theta \qquad for\ equilibrium$$

$$The\ post-buckling\ P-\theta\ relationship\ is\ given\ above$$



ENERGY METHOD – LARGE DEFLECTIONS

- Large deflection analysis
  - See the post-buckling load-displacement path shown below
  - The load carrying capacity decreases after buckling at P_{cr}
  - P_{cr} is where $\theta \rightarrow 0$



ENERGY METHOD – LARGE DEFLECTIONS

- Large deflection analysis – Examine the stability of equilibrium using higher order derivatives of Π

$$\Pi = \frac{1}{2} k\ L^2 \sin^2\theta - P\ L(1-\cos\theta)$$

$$\frac{d\Pi}{d\theta} = k\ L^2 \sin\theta\cos\theta - P\ L\sin\theta$$

$$\frac{d^2\Pi}{d\theta^2} = k\ L^2 \cos 2\theta - P\ L\cos\theta$$

$$For\ equilibrium\ P = k\ L\cos\theta$$

$$\therefore \frac{d^2\Pi}{d\theta^2} = k\ L^2 \cos 2\theta - k\ L^2 \cos^2\theta$$

$$\therefore \frac{d^2\Pi}{d\theta^2} = k\ L^2 (\cos^2\theta - \sin^2\theta) - k\ L^2 \cos^2\theta$$

$$\therefore \frac{d^2\Pi}{d\theta^2} = -k\ L^2 \sin^2\theta$$

$$\therefore \frac{d^2\Pi}{d\theta^2} < 0 \quad ALWAYS. \quad \underline{HENCE\ UNSTABLE}$$

ENERGY METHOD – LARGE DEFLECTIONS

- At θ =0, the second derivative of Π=0. Therefore, inconclusive.
- Consider the Taylor series expansion of Π at θ=0

$$\Pi = \Pi\big|_{\theta=0} + \frac{d\Pi}{d\theta}\bigg|_{\theta=0}\theta + \frac{1}{2!}\frac{d^2\Pi}{d\theta^2}\bigg|_{\theta=0}\theta^2 + \frac{1}{3!}\frac{d^3\Pi}{d\theta^3}\bigg|_{\theta=0}\theta^3 + \frac{1}{4!}\frac{d^4\Pi}{d\theta^4}\bigg|_{\theta=0}\theta^4 + \ldots\ldots + \frac{1}{n!}\frac{d^n\Pi}{d\theta^n}\bigg|_{\theta=0}\theta^n$$

- Determine the first non-zero term of Π,

$$\Pi = \frac{1}{2}k\,L^2\sin^2\theta - P\,L(1-\cos\theta) = 0$$

$$\frac{d\Pi}{d\theta} = \frac{1}{2}k\,L^2\sin 2\theta - P\,L\sin\theta = 0$$

$$\frac{d^2\Pi}{d\theta^2} = k\,L^2\cos 2\theta - P\,L\cos\theta = 0$$

$$\frac{d^3\Pi}{d\theta^3} = -2k\,L^2\sin 2\theta + P\,L\sin\theta = 0$$

$$\frac{d^4\Pi}{d\theta^4} = -4k\,L^2\cos 2\theta + P\,L\cos\theta$$

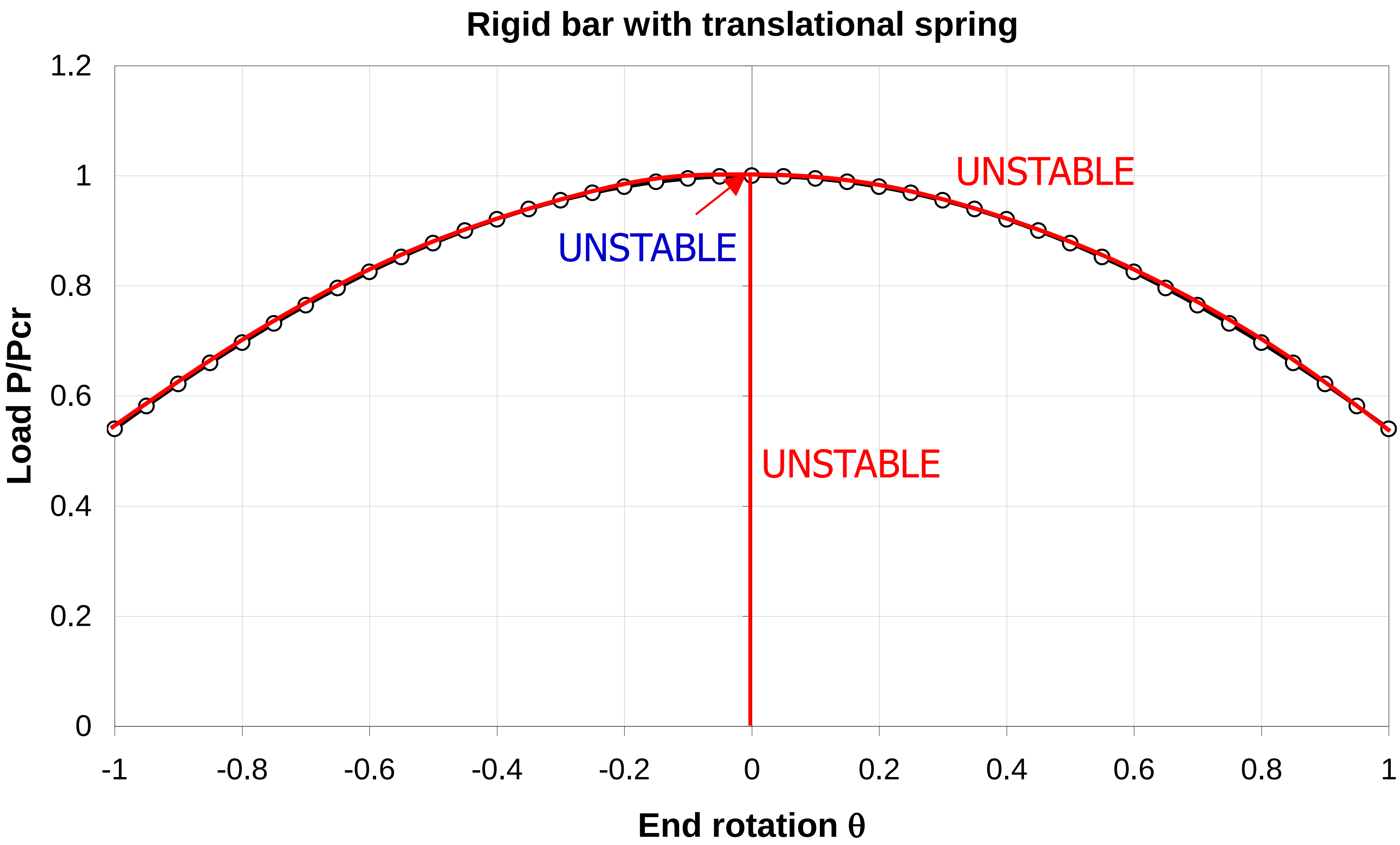
$$\therefore \frac{d^4\Pi}{d\theta^4} = -4k\,L^2 + k\,L^2 = -3k\,L^2$$

$$\therefore \frac{d^4\Pi}{d\theta^4} < 0$$

$$\therefore UNSTABLE\ at\ \theta = 0\ when\ buckling\ occurs$$

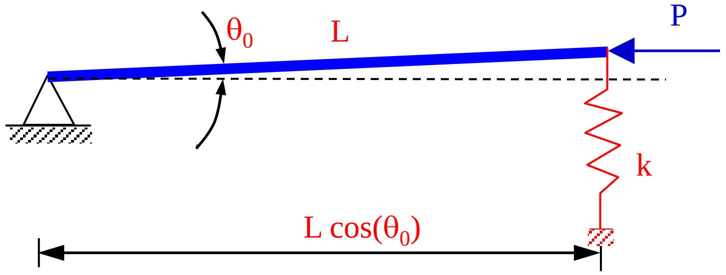
- Since the first non-zero term is < 0, the state is unstable at $P=P_{cr}$ and θ=

ENERGY METHOD – LARGE DEFLECTIONS

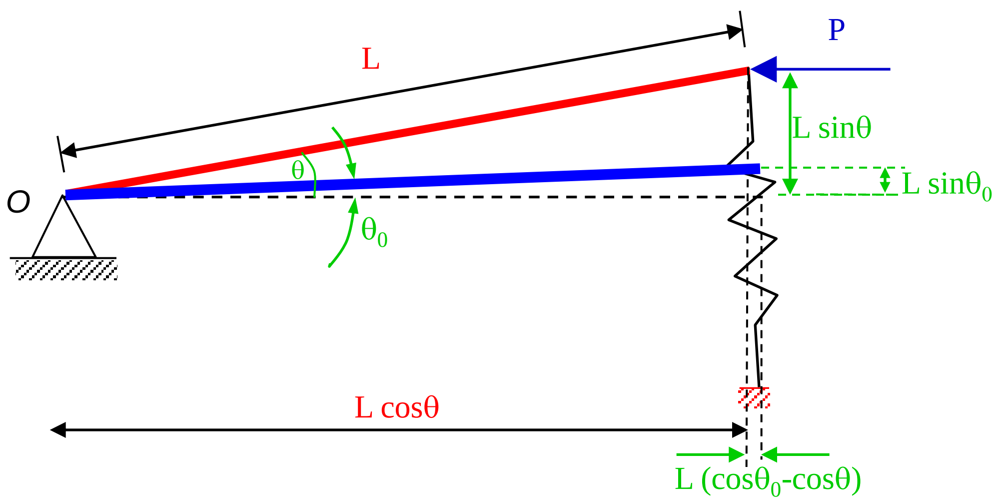


ENERGY METHOD - IMPERFECTIONS

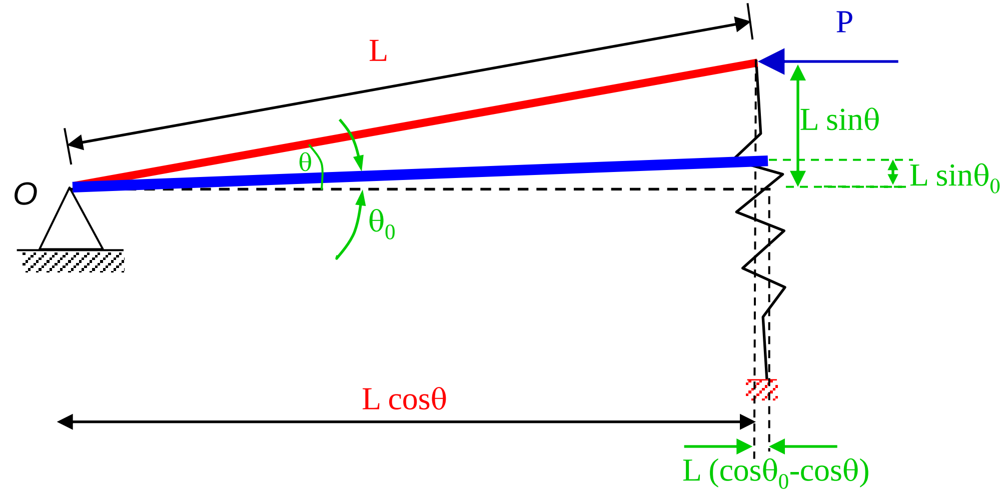
- Consider example 2 – but as a system with imperfections
  - The initial imperfection given by the angle θ_0 as shown below



  - The free body diagram of the deformed system is shown below



ENERGY METHOD - IMPERFECTIONS



$$\prod = U - W_e$$

$$U = \frac{1}{2} k\ L^2\ (\sin\theta - \sin\theta_0)^2$$

$$W_e = P\ L\,(\cos\theta_0 - \cos\theta)$$

$$\prod = \frac{1}{2} k\ L^2 (\sin\theta - \sin\theta_0)^2 - P\ L\,(\cos\theta_0 - \cos\theta)$$

$$\frac{d\prod}{d\theta} = k\ L^2 (\sin\theta - \sin\theta_0)\cos\theta - P\ L\sin\theta$$

$$For\ equilibrium;\ \frac{d\prod}{d\theta} = 0$$

$$Therefore, \qquad k\ L^2 (\sin\theta - \sin\theta_0)\cos\theta - P\ L\sin\theta = 0$$

$$Therefore, \qquad P = k\ L\cos\theta\,(1 - \frac{\sin\theta_0}{\sin\theta}) \quad for\ equilibrium$$

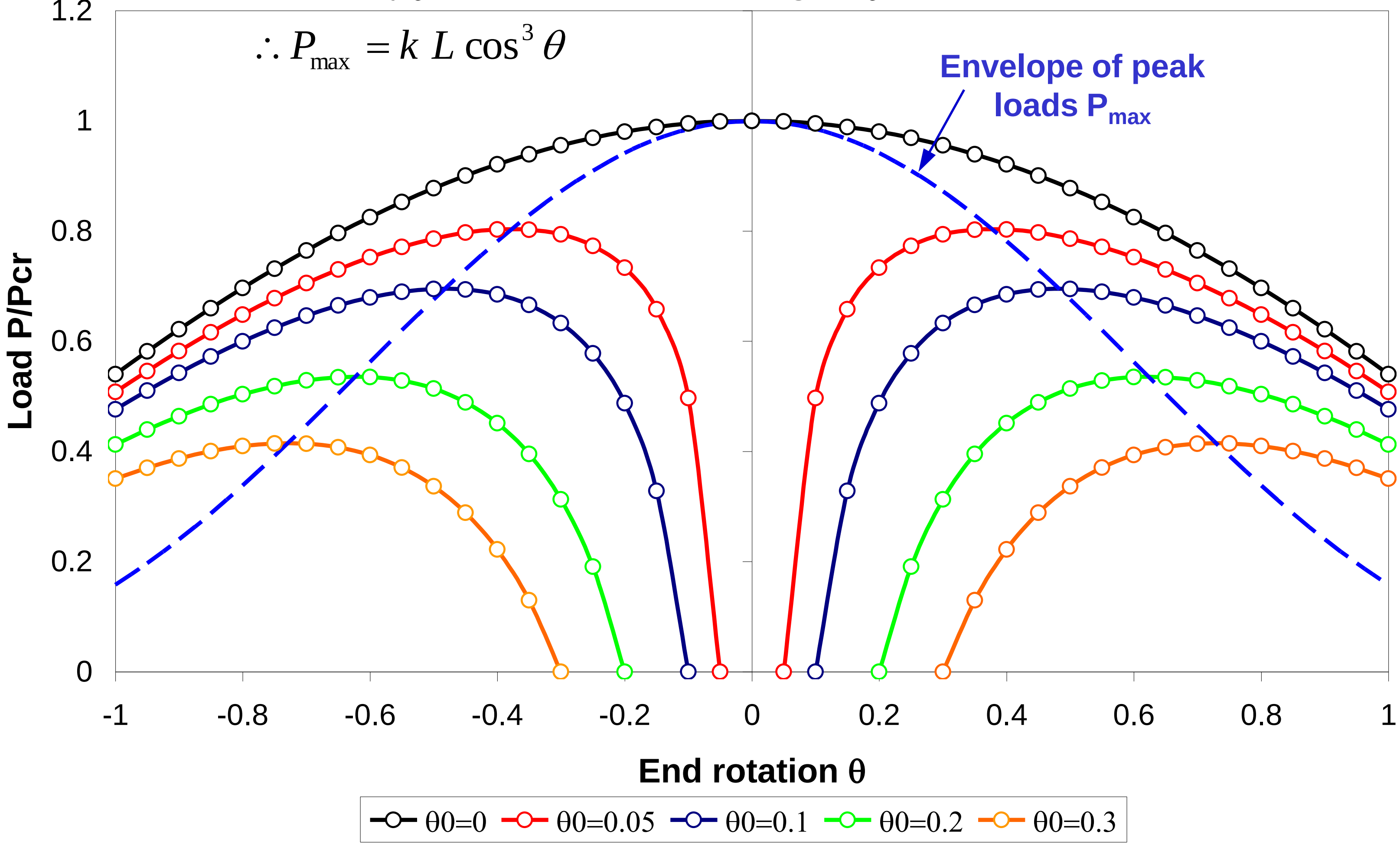
The equilibrium $P - \theta$ relationship is given above

ENERGY METHOD - IMPERFECTIONS

$$P = k\ L \cos\theta \,(1 - \frac{\sin\theta_0}{\sin\theta}) \qquad \therefore \frac{P}{P_{cr}} = \cos\theta\,(1 - \frac{\sin\theta_0}{\sin\theta})$$

$$P_{\max} \Rightarrow \frac{dP}{d\theta} = 0 \quad \therefore k\ L(-\sin\theta + \frac{\sin\theta_0}{\sin^2\theta}) = 0 \quad \therefore \sin\theta_0 = \sin^3\theta$$

$$\therefore P_{\max} = k\ L \cos^3\theta$$



ENERGY METHOD - IMPERFECTIONS

- As shown in the figure, deflection starts as soon as loads are applied. There is no bifurcation of load-deformation path for imperfect systems. The load-deformation path remains in the same state through-out.
- The smaller the imperfection magnitude, the close the load-deformation paths to the perfect system load –deformation path.
- The magnitude of load, is influenced significantly by the imperfection magnitude.
- All real systems have imperfections. They may be very small but will be there
- The magnitude of imperfection is not easy to know or guess. Hence if a perfect system analysis is done, the results will be close for an imperfect system with small imperfections.
- However, for an unstable system – the effects of imperfections may be too large.

ENERGY METHODS – IMPERFECT SYSTEMS

- Examine the stability of the imperfect system using higher order derivatives of Π

$$\Pi = \frac{1}{2} k\, L^2\, (\sin\theta - \sin\theta_0)^2 - P\, L\,(\cos\theta_0 - \cos\theta)$$

$$\frac{d\Pi}{d\theta} = k\, L^2\, (\sin\theta - \sin\theta_0)\cos\theta - P\, L \sin\theta$$

$$\frac{d^2\Pi}{d\theta^2} = k\, L^2\, (\cos 2\theta + \sin\theta_0 \sin\theta) - P\, L\cos\theta$$

$$\textit{For equilibrium } P = k\, L\left(1 - \frac{\sin\theta_0}{\sin\theta}\right)$$

$$\therefore \frac{d^2\Pi}{d\theta^2} = k\, L^2\, (\cos 2\theta + \sin\theta_0 \sin\theta) - k\, L^2\left(1 - \frac{\sin\theta_0}{\sin\theta}\right)\cos^2\theta$$

$$\therefore \frac{d^2\Pi}{d\theta^2} = k\, L^2\left[\cos^2\theta - \sin^2\theta + \sin\theta_0 \sin\theta - \cos^2\theta + \frac{\sin\theta_0 \cos^2\theta}{\sin\theta}\right]$$

$$\therefore \frac{d^2\Pi}{d\theta^2} = k\, L^2\left[-\sin^2\theta + \sin\theta_0 \sin\theta + \frac{\sin\theta_0 \cos^2\theta}{\sin\theta}\right]$$

$$\therefore \frac{d^2\Pi}{d\theta^2} = k\, L^2\left[\frac{-\sin^3\theta + \sin\theta_0(\sin^2\theta + \cos^2\theta)}{\sin\theta}\right]$$

$$\therefore \frac{d^2\Pi}{d\theta^2} = k\, L^2\left[\frac{-\sin^3\theta + \sin\theta_0}{\sin\theta}\right]$$

ENERGY METHOD – IMPERFECT SYSTEMS

$$\frac{d^2\Pi}{d\theta^2} = k L^2 \left[\frac{-\sin^3\theta + \sin\theta_0}{\sin\theta}\right]$$

$$\frac{d^2\Pi}{d\theta^2} > 0 \; when \; P < P_{\max} \quad \therefore Stable$$

$$\frac{d^2\Pi}{d\theta^2} < 0 \; when \; P > P_{\max} \quad \therefore Unstable$$

$$P = k\,L\cos\theta\,(1 - \frac{\sin\theta_0}{\sin\theta}) \qquad and \qquad P_{\max} = k\,L\cos^3\theta$$

$When \; P < P_{\max}$

$$k\,L\cos\theta\,(1 - \frac{\sin\theta_0}{\sin\theta}) < k\,L\cos^3\theta$$

$$\therefore 1 - \frac{\sin\theta_0}{\sin\theta} < \cos^2\theta$$

$$\therefore 1 - \frac{\sin\theta_0}{\sin\theta} < 1 - \sin^2\theta$$

$$\therefore \sin\theta_0 > \sin^3\theta \qquad and \qquad \frac{d^2\Pi}{d\theta^2} = k\,L^2\left[\frac{\sin\theta_0 - \sin^3\theta}{\sin\theta}\right] > 0$$

$When \; P > P_{\max}$

$$k\,L\cos\theta\,(1 - \frac{\sin\theta_0}{\sin\theta}) > k\,L\cos^3\theta$$

$$\therefore 1 - \frac{\sin\theta_0}{\sin\theta} > \cos^2\theta$$

$$\therefore 1 - \frac{\sin\theta_0}{\sin\theta} > 1 - \sin^2\theta$$

$$\therefore \sin\theta_0 < \sin^3\theta \qquad and \qquad \frac{d^2\Pi}{d\theta^2} = k\,L^2\left[\frac{\sin\theta_0 - \sin^3\theta}{\sin\theta}\right] < 0$$

Chapter 2. – Second-Order Differential Equations

- This chapter focuses on deriving second-order differential equations governing the behavior of elastic members
- 2.1 – First order differential equations
- 2.2 – Second-order differential equations

2.1 First-Order Differential Equations

- Governing the behavior of structural members
  - Elastic, Homogenous, and Isotropic
  - Strains and deformations are really small – small deflection theory
  - Equations of equilibrium in undeformed state
- Consider the behavior of a beam subjected to bending and axial forces

2.1 First-Order Differential Equations

- Assume tensile forces are positive and moments are positive according to the right-hand rule
- Longitudinal stress due to bending

$$\sigma = \frac{P}{A} + \frac{M_x}{I_x} y - \frac{M_y}{I_y} x$$

- This is true when the x-y axis system is

a centroidal and principal axis system.

$$\int_A y\ dA = \int_A x\ dA = \int_A x\ y\ dA = 0 \quad \therefore Centroidal\ \ axis$$

$$\int_A dA = A; \quad \int_A x^2\ dA = I_y; \quad \int_A y^2\ dA = I_x$$

$$I_x\ \ and\ \ I_y\ \ are\ \ principal\ \ moment\ of\ \ inertia$$

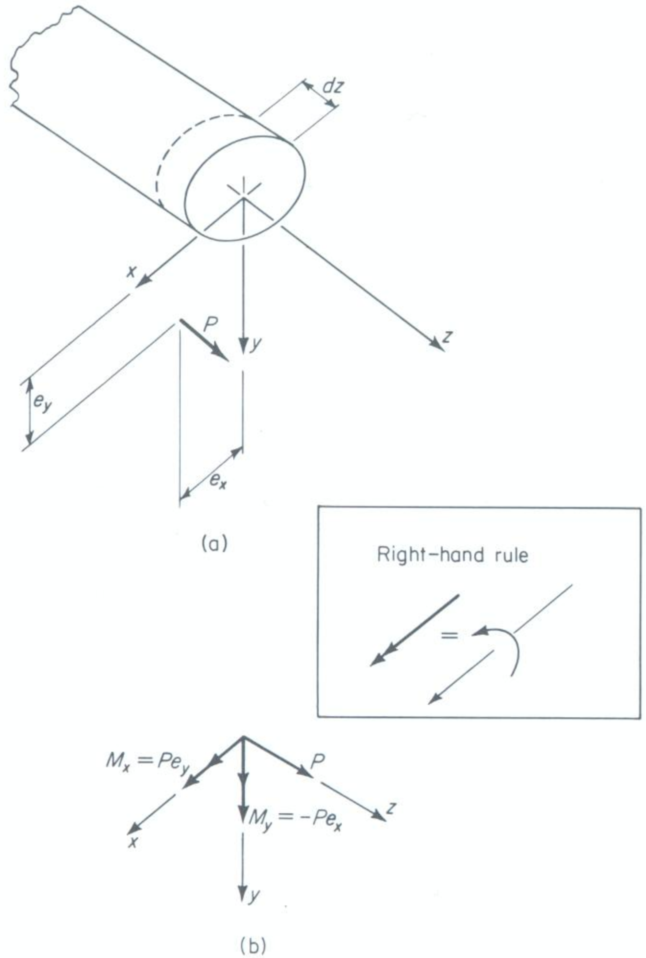


Fig. 2.1. Cross section of a bar subjected to bending and axial force

2.1 First-Order Differential Equations

- The corresponding strain is $\varepsilon = \frac{P}{A\,E} + \frac{M_x}{E\,I_x} y - \frac{M_y}{E\,I_y} x$
- If $P = M_y = 0$, then $\varepsilon = \frac{M_x}{E\,I_x} y$
- Plane-sections remain plane and perpendicular to centroidal axis before and after bending
- The measure of bending is curvature ϕ which denotes the change in the slope of the centroidal axis between two point *dz* apart

$$\tan\phi_y = \frac{\varepsilon}{y}$$

$$For\ small\ deformations\ \tan\phi_y \cong \phi_y$$

$$\therefore \phi_y = \frac{\varepsilon}{y}$$

$$\therefore \phi_y = \frac{M_x}{E\,I_x}$$

$$\therefore M_x = E\,I_x\,\phi_y \quad and\ similarly\ M_y = E\,I_y\,\phi_x$$

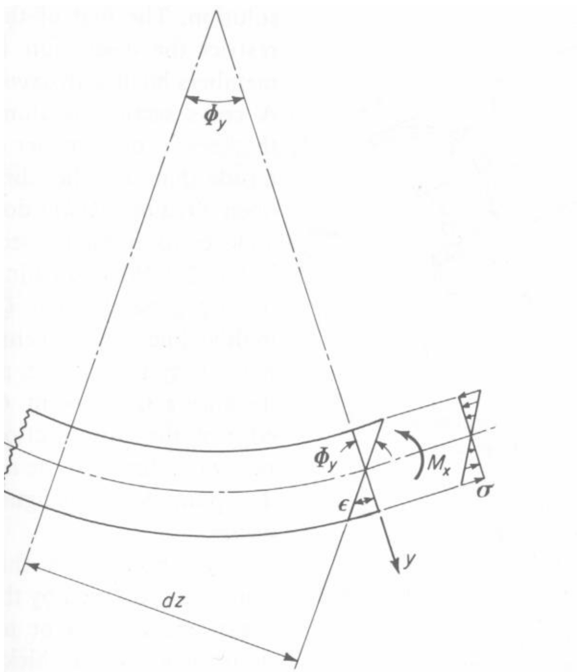
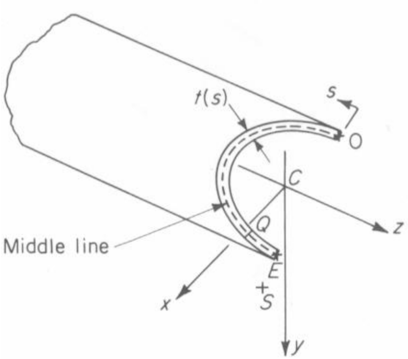


Fig. 2.2. Curvature, strain, and stress due to bending

2.1 First-Order Differential Equations

- Shear Stresses due to bending



$O\ (x_1, y_1)$	Origin of reference s
$E\ (x_2, y_2)$	End of reference s
$C\ (0, 0)$	Centroid
$Q\ (x, y)$	General point
$S\ (x_0, y_0)$	Shear center
$t(s)$	Thickness, function of s
s	Coordinate along middle line of cross section
x, y	Principal centroidal axes
z	Longitudinal centroidal axis

Fig. 2.3. Dimensions of a thin-walled open cross section

$$\tau\, t = -\frac{V_y}{I_x}\int_O^s y\, t\, ds$$

$$\tau\, t = -\frac{V_x}{I_y}\int_O^s x\, t\, ds$$

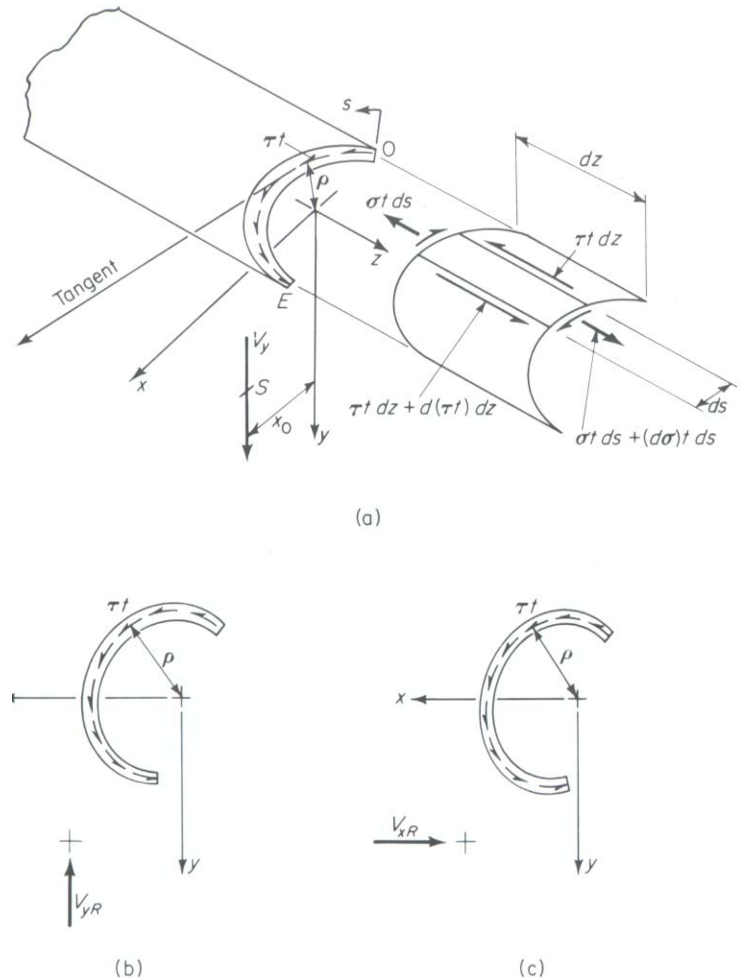


Fig. 2.4. Shear stresses on an element of a thin-walled open cross section

2.1 First-Order Differential Equations

- Differential equations of bending
- Assume principle of superposition
  - Treat forces and deformations in y-z and x-z plane seperately
  - Both the end shears and q_y act in a plane parallel to the y-z plane through the shear center S

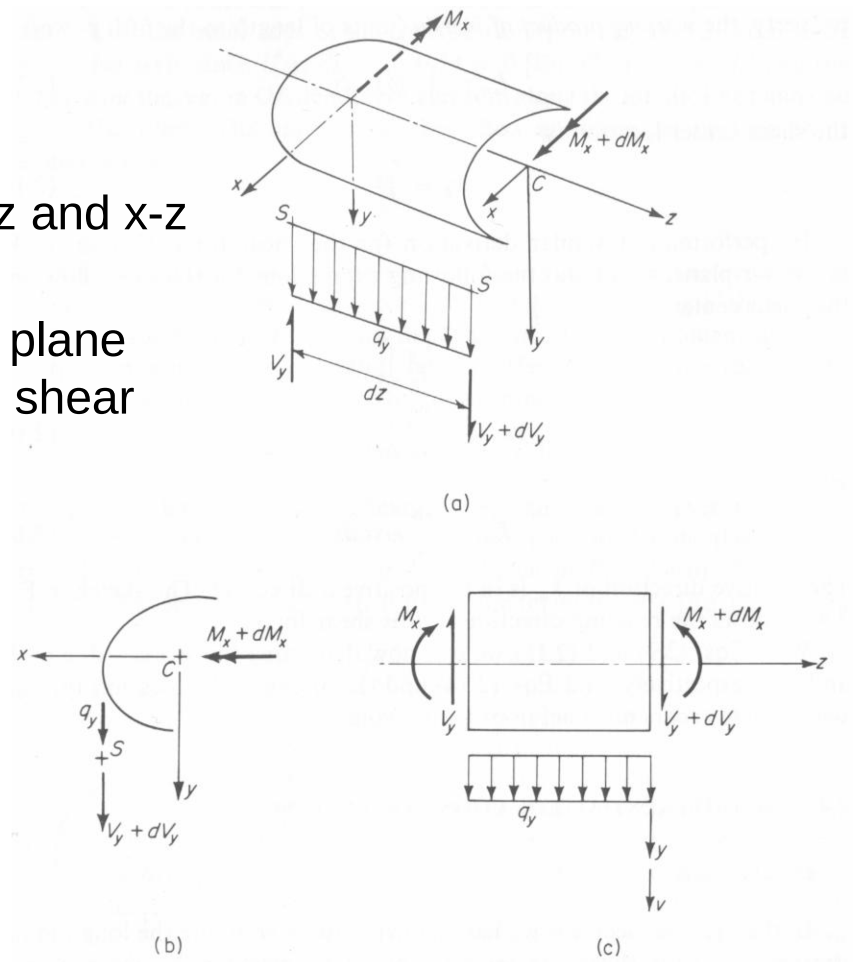


Fig. 2.6. Forces in the *y-z* plane of a bar element

$$\frac{dV_y}{dz} = -q_y$$

$$\frac{dM_x}{dz} = V_y$$

$$\therefore \frac{d^2 M_x}{dz^2} = -q_y$$

$$\therefore \frac{d^2(E\, I_x\, \phi_y)}{dz^2} = -q_y$$

$$\therefore E\, I_x\, \phi_y'' = -q_y$$

2.1 First-Order Differential Equations

- Differential equations of bending

$$E\,I_x\,\phi_y'' = -q_y$$

$$\phi_y = -\frac{v''}{\left[1+(v')^2\right]^{3/2}}$$

For small deflections

$$\phi_y = -v''$$

$$\therefore E\,I_x\,v^{iv} = q_y$$

$$\text{Similarly } E\,I_y\,u^{iv} = q_x$$

$$u \rightarrow \text{deflection in positive } x \text{ direction}$$

$$v \rightarrow \text{deflection in positive } y \text{ direction}$$

- Fourth-order differential equations using first-order force-deformation theory

Torsion behavior – Pure and Warping Torsion

- Torsion behavior – uncoupled from bending behavior
- Thin walled open cross-section subjected to torsional moment
  - This moment will cause twisting and warping of the cross-section.
  - The cross-section will undergo pure and warping torsion behavior.
  - Pure torsion will produce only shear stresses in the section
  - Warping torsion will produce both longitudinal and shear stresses
  - The internal moment produced by the pure torsion response will be equal to M_{sv} and the internal moment produced by the warping torsion response will be equal to M_w.
  - The external moment will be equilibriated by the produced internal moments
- $M_Z = M_{SV} + M_W$

Pure and Warping Torsion

$M_Z = M_{SV} + M_W$

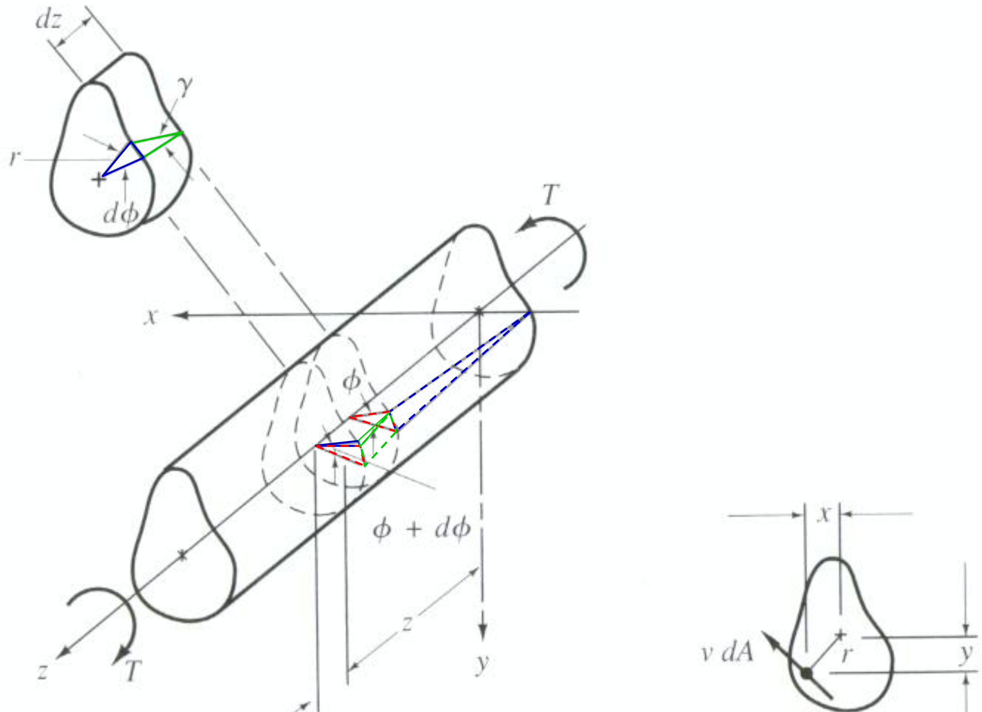
Where,

- $M_{SV} = G\,K_T\,\phi'$ and $M_W = -\,E\,I_w\,\phi'''$
- M_{SV} = Pure or Saint Venant's torsion moment
- $K_T = J$ = Torsional constant =
- ϕ is the angle of twist of the cross-section. It is a function of z.
- I_W is the warping moment of inertia of the cross-section. This is a new cross-sectional property you may not have seen before.

$M_Z = G\,K_T\,\phi' - E\,I_w\,\phi'''$ ……… (3), differential equation of torsion

Pure Torsion Differential Equation

- Lets look closely at pure or Saint Venant's torsion. This occurs when the warping of the cross-section is unrestrained or absent



$$\gamma\, dz = r\, d\phi$$

$$\therefore \gamma = r\frac{d\phi}{dz} = r\,\phi'$$

$$\therefore \tau = G\, r\, \phi'$$

$$\therefore M_{SV} = \int_A \tau\, r\, dA = G\,\phi' \int_A r^2\, dA$$

$$\therefore \boxed{M_{SV} = G\, K_T\, \phi'}$$

$$where,\ K_T = J = \int_A r^2\, dA$$

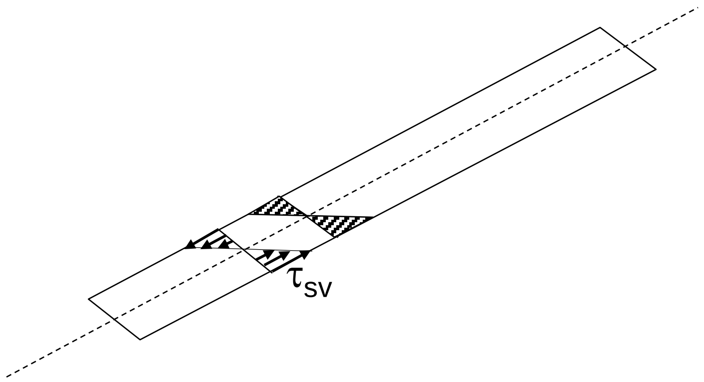
- For a circular cross-section – warping is absent. For thin-walled open cross-sections, warping will occur.
- The out of plane warping deformation w can be calculated using an equation I will not show.

Pure Torsion Stresses

The torsional shear stresses vary linearly about the center of the thin plate

$$\tau_{SV} = G\ r\ \phi'$$

$$\left(\tau_{SV}\right)_{\max} = G\ t\ \phi'$$



Warping deformations

- The warping produced by pure torsion can be restrained by the: (a) end conditions, or (b) variation in the applied torsional moment (non-uniform moment)
- The restraint to out-of-plane warping deformations will produce longitudinal stresses (σ_w) , and their variation along the length will produce warping shear stresses (τ_w) .

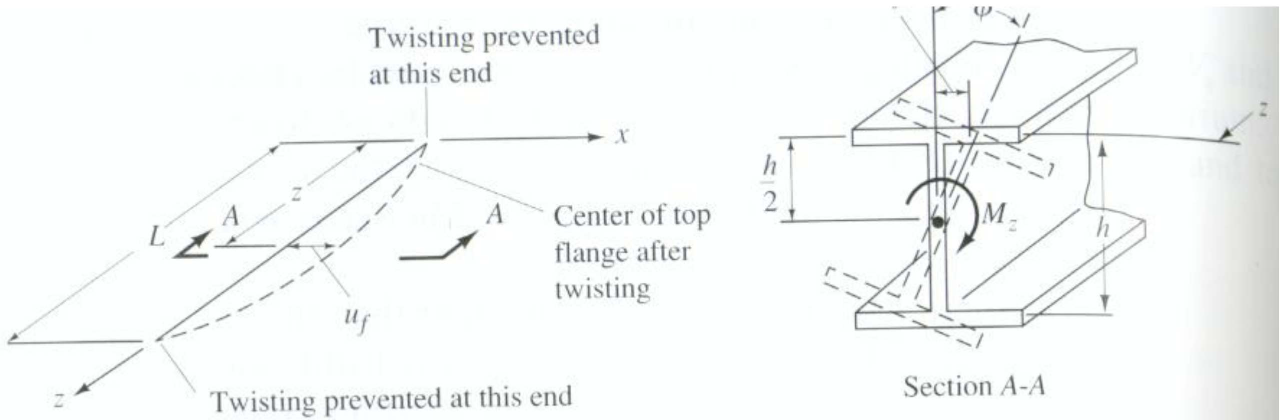


Figure 8.5.2 Torsion of an I-shaped section.

Warping Torsion Differential Equation

- Lets take a look at an approximate derivation of the warping torsion differential equation.
  - This is valid only for I and C shaped sections.

$$u_f = \phi \frac{h}{2}$$

$$\text{where } u_f = \text{flange lateral displacement}$$

$$M_f = \text{moment in the flange}$$

$$V_f = \text{Shear force in the flange}$$

$$E\, I_f\, u_f'' = -M_f \quad \cdots\cdots\cdots \text{borrowing d.e. of bending}$$

$$E\, I_f\, u_f''' = -V_f$$

$$M_W = V_f\, h$$

$$\therefore M_W = -E\, I_f\, u_f'''\, h$$

$$\therefore M_W = -E\, I_f\, \frac{h^2}{2} \phi'''$$

$$\therefore M_W = -E\, I_W\, \phi'''$$

$$\text{where } I_W \text{ is warping moment of inertia} \rightarrow \text{new section property}$$

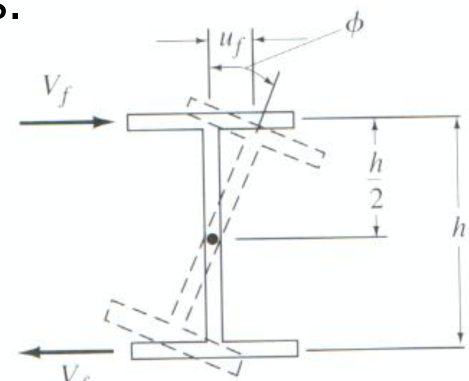


Figure 8.5.3 Warping shear force on I-shaped section.

Torsion Differential Equation Solution

- Torsion differential equation $M_Z = M_{SV} + M_W = G\, K_T\, \phi' - E\, I_W\, \phi'''$
- This differential equation is for the case of concentrated torque

$$G\, K_T\, \phi' - E\, I_w\, \phi''' = M_Z$$

$$\therefore \phi''' - \frac{G\, K_T}{E\, I_W}\phi' = -\frac{M_Z}{E\, I_W}$$

$$\therefore \phi''' - \lambda^2\, \phi' = -\frac{M_Z}{E\, I_W}$$

$$\therefore \phi = C_1 + C_2 \cosh \lambda z + C_3 \sinh \lambda z + \frac{M_z\, z}{\lambda^2\, E\, I_W}$$

- Torsion differential equation for the case of distributed torque

$$m_Z = -\frac{dM_Z}{dz}$$

$$G\, K_T\, \phi'' - E\, I_w\, \phi^{iv} = -m_Z$$

$$\therefore \phi^{iv} - \frac{G\, K_T}{E\, I_W}\phi'' = \frac{m_Z}{E\, I_W}$$

$$\therefore \phi^{iv} - \lambda^2\, \phi'' = \frac{m_Z}{E\, I_W}$$

$$\therefore \phi = C_4 + C_5\, z + C_6 \cosh \lambda z + C_7 \sinh \lambda z - \frac{m_z\, z^2}{2\, G\, K_T}$$

- The coefficients C_1 .... C_6 can be obtained using end conditions

Torsion Differential Equation Solution

- Torsionally fixed end conditions are given by $\phi = \phi' = 0$
- These imply that twisting and warping at the fixed end are fully restrained. Therefore, equal to zero.
- Torsionally pinned or simply-supported end conditions given by:

  $$\phi = \phi'' = 0$$

- These imply that at the pinned end twisting is fully restrained (ϕ=0) and warping is unrestrained or free. Therefore, σ_W=0 → ϕ''=0
- Torsionally free end conditions given by $\phi' = \phi'' = \phi''' = 0$
- These imply that at the free end, the section is free to warp and there are no warping normal or shear stresses.
- Results for various torsional loading conditions given in the AISC Design Guide 9 – can be obtained from my private site

Warping Torsion Stresses

- Restraint to warping produces longitudinal and shear stresses

$$\sigma_W = E\, W_n\, \phi''$$

$$\tau_W\, t = -E\, S_W \phi'''$$

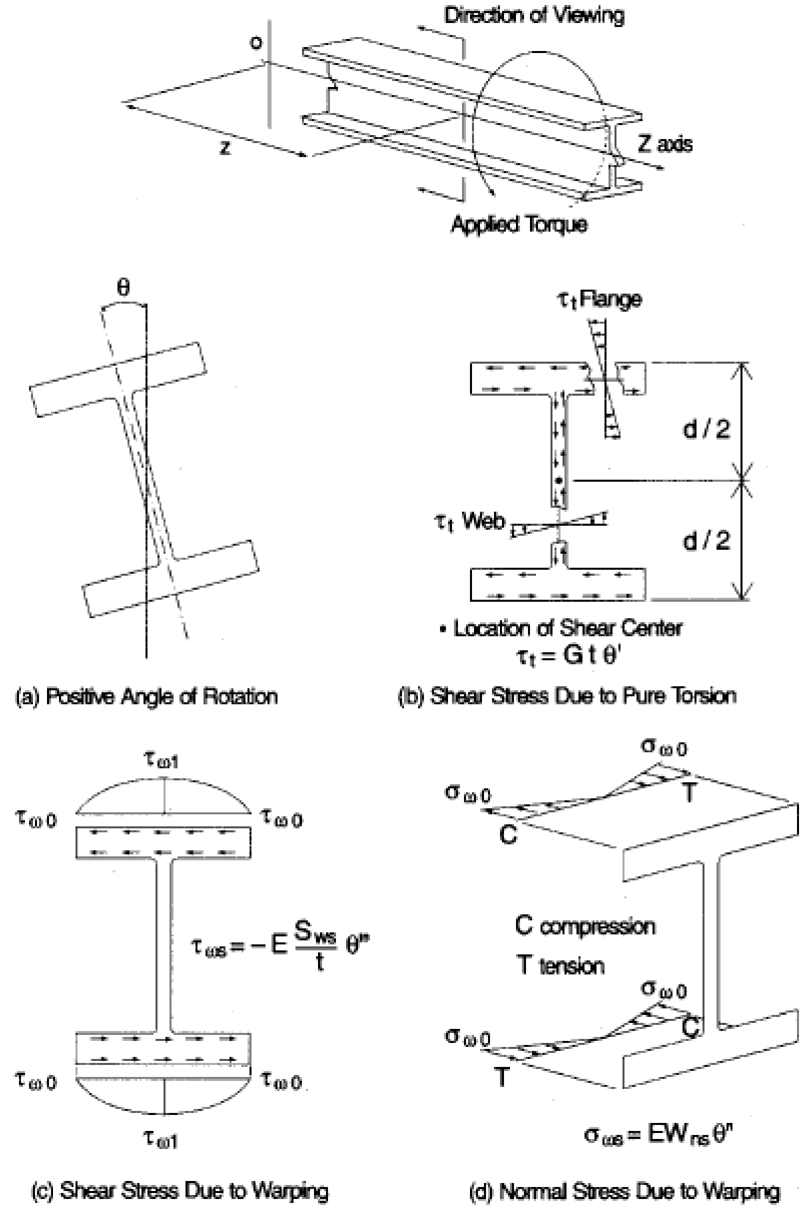
where,

$$W_n = Normalized\ Unit\ Warping - Section\ Property$$

$$S_W = Warping\ Statical\ \ Moment - Section\ Property$$

- The variation of these stresses over the section is defined by the section property W_n and S_w
- The variation of these stresses along the length of the beam is defined by the derivatives of ϕ.
- *Note that a major difference between bending and torsional behavior is*
  - *The stress variation along length for torsion is defined by derivatives of ϕ, which cannot be obtained using force equilibrium.*
  - *The stress variation along length for bending is defined by derivatives of v, which can be obtained using force equilibrium (M, V diagrams).*

Torsional Stresses



Torsional Stresses

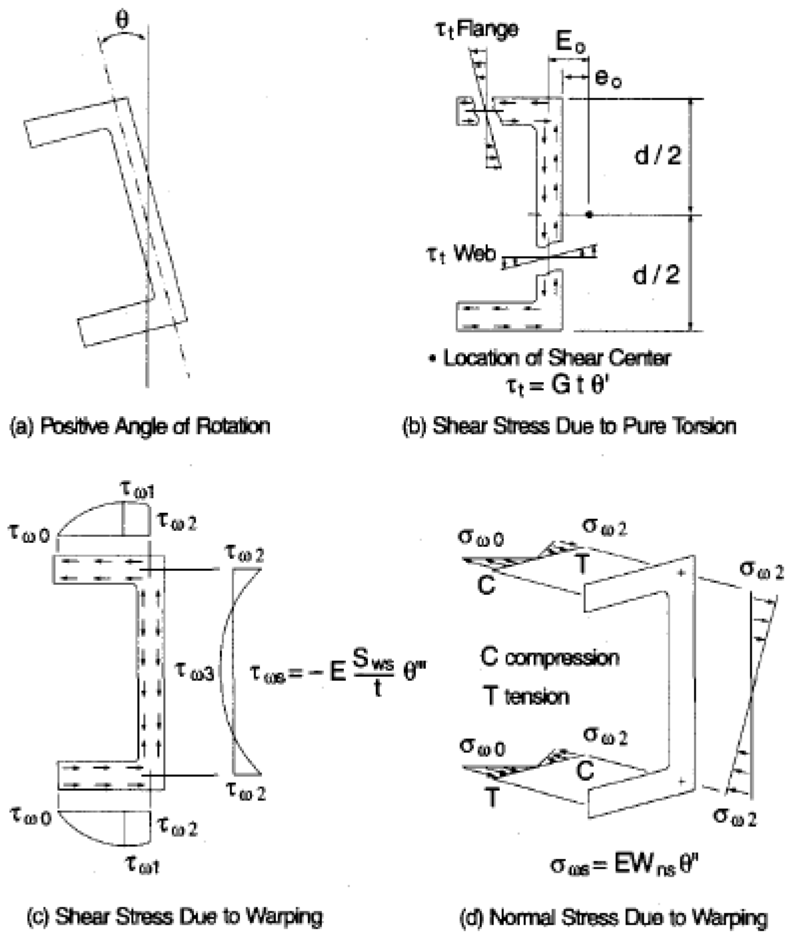


Figure 4.2.

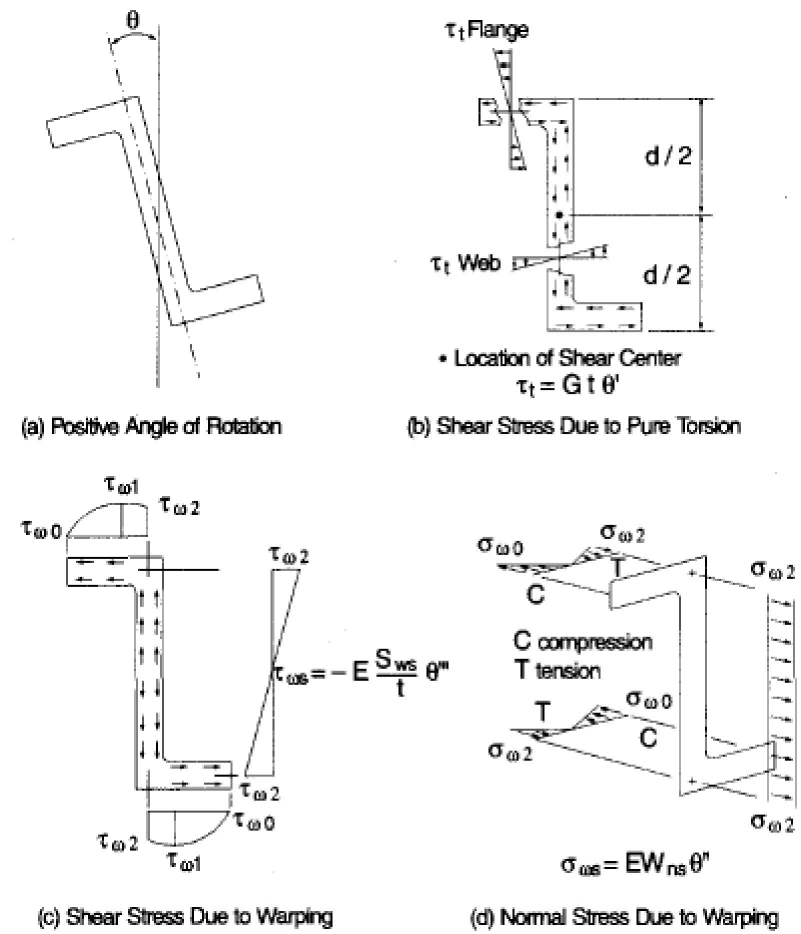
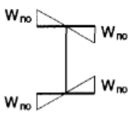
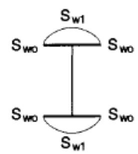


Figure 4.3.

Torsional Section Properties for I and C Shapes

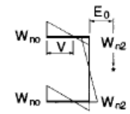
W-, M-, S-, and HP-Shapes

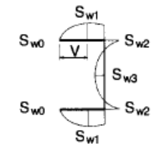




Shape	Torsional Properties					Statical Moments	
	J	C_w	a	W_{no}	S_{w1}	Q_f	Q_w
	in.^{4}	in.^{6}	in.	in.^{2}	in.^{4}	in.^{3}	in.^{3}
W21×93	6.03	9,940	65.3	43.6	85.3	38.2	110
83	4.34	8,630	71.8	43.0	75.0	34.2	98.0
73	3.02	7,410	79.7	42.5	65.2	30.3	86.2
68	2.45	6,760	84.5	42.3	59.9	28.0	79.9
62	1.83	5,960	91.8	42.0	53.2	25.1	72.2
W21×57	1.77	3,190	68.3	33.4	35.6	20.9	64.3
50	1.14	2,570	76.4	33.1	28.9	17.2	55.0
44	0.77	2,110	84.2	32.8	24.0	14.5	47.7
W18×311	177	75,700	33.3	59.0	483	141	376
283	135	65,600	35.5	57.5	427	127	338
258	104	57,400	37.8	56.4	382	116	306
234	79.7	49,900	40.3	55.2	339	105	274
211	59.3	43,200	43.4	54.2	299	94.3	245
192	45.2	37,900	46.6	53.3	267	85.7	221
175	34.2	33,200	50.1	52.5	237	77.2	199
158	25.4	28,900	54.3	51.6	210	69.4	178
143	19.4	25,700	58.6	51.0	189	63.2	161
130	14.7	22,700	63.2	50.4	169	57.1	145
W18×119	10.6	20,300	70.4	50.4	151	50.6	131

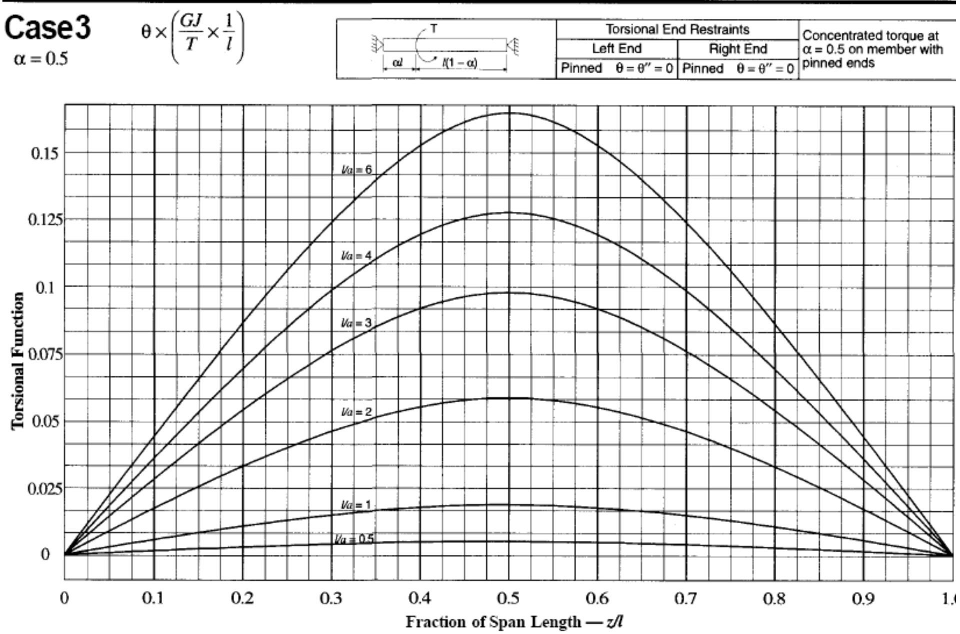
C- and MC-Shapes

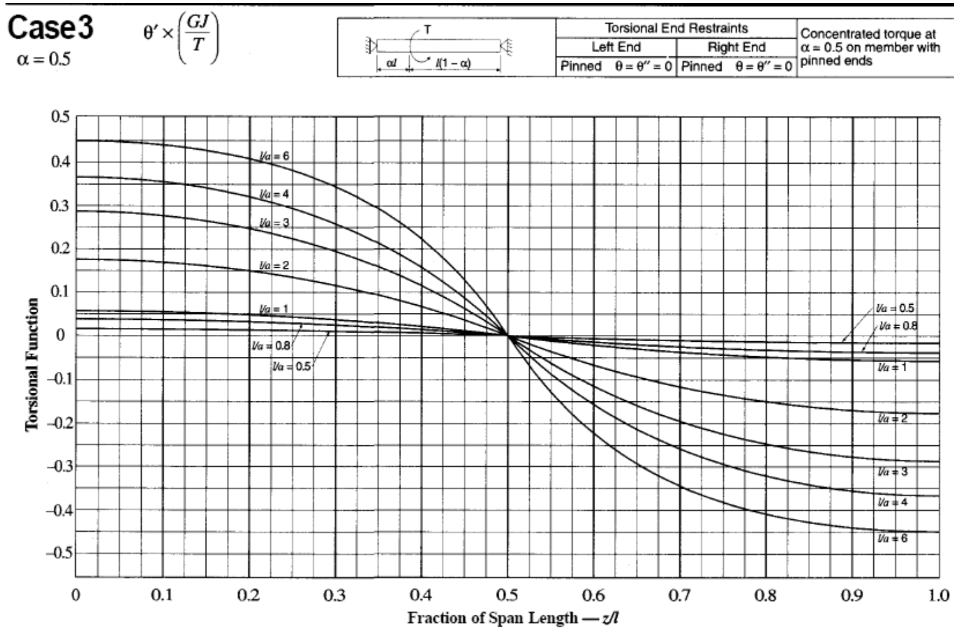


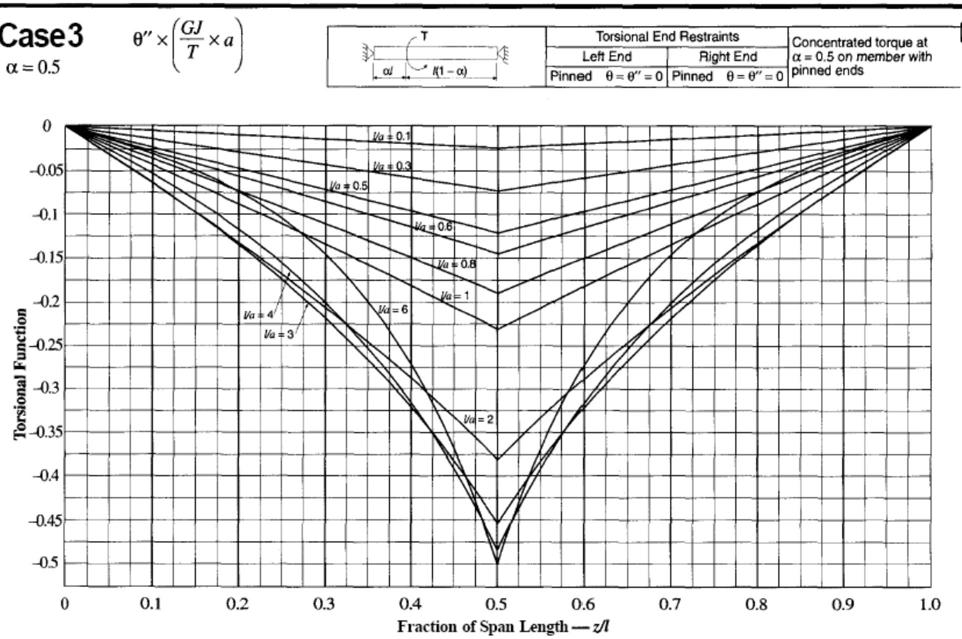


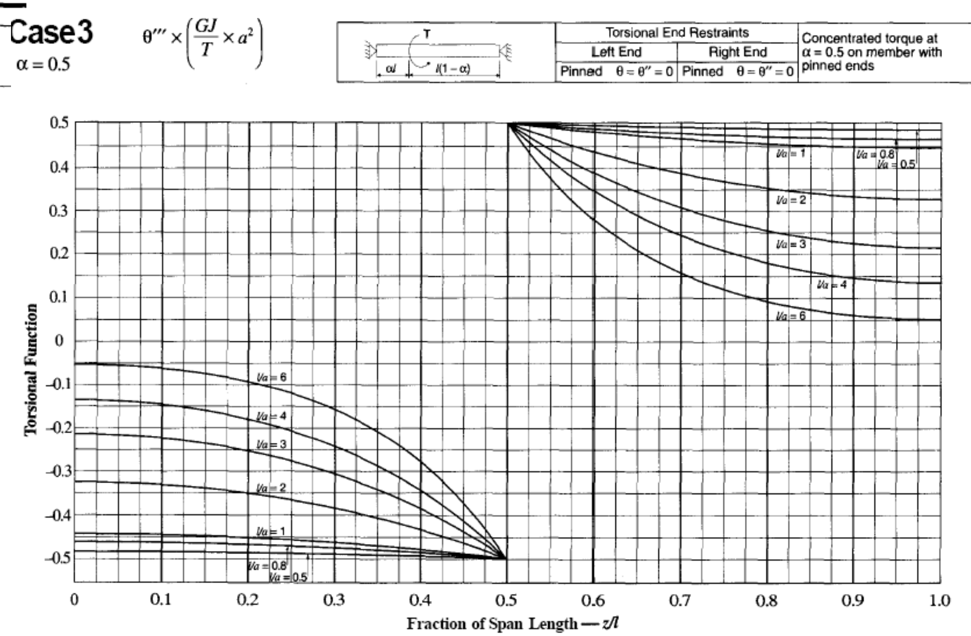
Shape	Torsional Properties									Statical Moments	
	J	C_w	a	W_{no}	W_{n2}	S_{w1}	S_{w2}	S_{w3}	E_o	Q_f	Q_w
	in.^{4}	in.^{6}	in.	in.^{2}	in.^{2}	in.^{4}	in.^{4}	in.^{4}	in.	in.^{3}	in.^{3}
MC18×58	2.81	1,070	31.4	24.4	9.08	21.4	18.4	9.21	1.05	19.7	48.0
51.9	2.03	986	35.5	23.5	9.53	19.8	16.6	8.27	1.10	19.7	44.0
45.8	1.45	897	40.0	22.5	10.1	18.2	14.6	7.29	1.16	19.7	39.9
42.7	1.23	852	42.4	22.0	10.4	17.4	13.5	6.75	1.19	19.7	37.9
MC13×50	2.98	558	22.0	17.4	7.49	14.9	12.2	6.09	1.21	14.0	30.6
40	1.57	463	27.6	16.1	8.12	12.7	9.48	4.60	1.31	14.0	25.8
35	1.14	413	30.6	15.3	8.57	11.5	7.86	4.00	1.38	14.0	23.4
31.8	0.94	380	32.4	14.8	8.84	10.7	6.90	3.37	1.43	14.0	21.9
MC12×50	3.24	411	18.1	14.5	6.55	12.9	10.3	5.14	1.16	13.3	28.4
45	2.35	374	20.3	13.9	6.78	11.9	9.08	4.56	1.20	13.3	26.1
40	1.70	336	22.6	13.3	7.05	10.9	7.83	3.92	1.25	13.3	23.9
35	1.25	297	24.8	12.6	7.36	9.83	6.47	3.24	1.30	13.3	21.7
31	1.01	268	26.2	12.0	7.71	8.89	5.20	2.86	1.37	13.3	21.6
MC12×10.6	0.06	11.7	22.5	6.00	2.22	0.95	0.82	0.41	0.379	2.61	6.36
MC10×41.1	2.27	270	17.5	12.5	5.95	9.59	7.44	3.72	1.26	9.86	19.8
33.6	1.21	224	21.9	11.6	6.35	8.23	5.77	2.83	1.35	9.86	17.0
28.5	0.79	194	25.2	10.9	6.70	7.26	4.52	2.19	1.42	9.86	15.2
MC10×25	0.64	125	22.5	9.40	5.75	5.39	3.38	1.77	1.22	7.66	13.0
22	0.51	111	23.7	8.93	6.01	4.87	2.66	1.44	1.28	7.66	11.7

ϕ and derivatives for concentrated torque at midspan









Summary of first order differential equations

$$-E\,I_x\,v'' = M_x \qquad \cdots\cdots\cdots(1)$$

$$E\,I_y\,u'' = M_y \qquad \cdots\cdots\cdots(2)$$

$$G\,K_T\phi' - E\,I_W\phi''' = M_z \qquad \cdots\cdots\cdots(3)$$

NOTES:

(1) Three uncoupled differential equations

(2) Elastic material – first order force-deformation theory

(3) Small deflections only

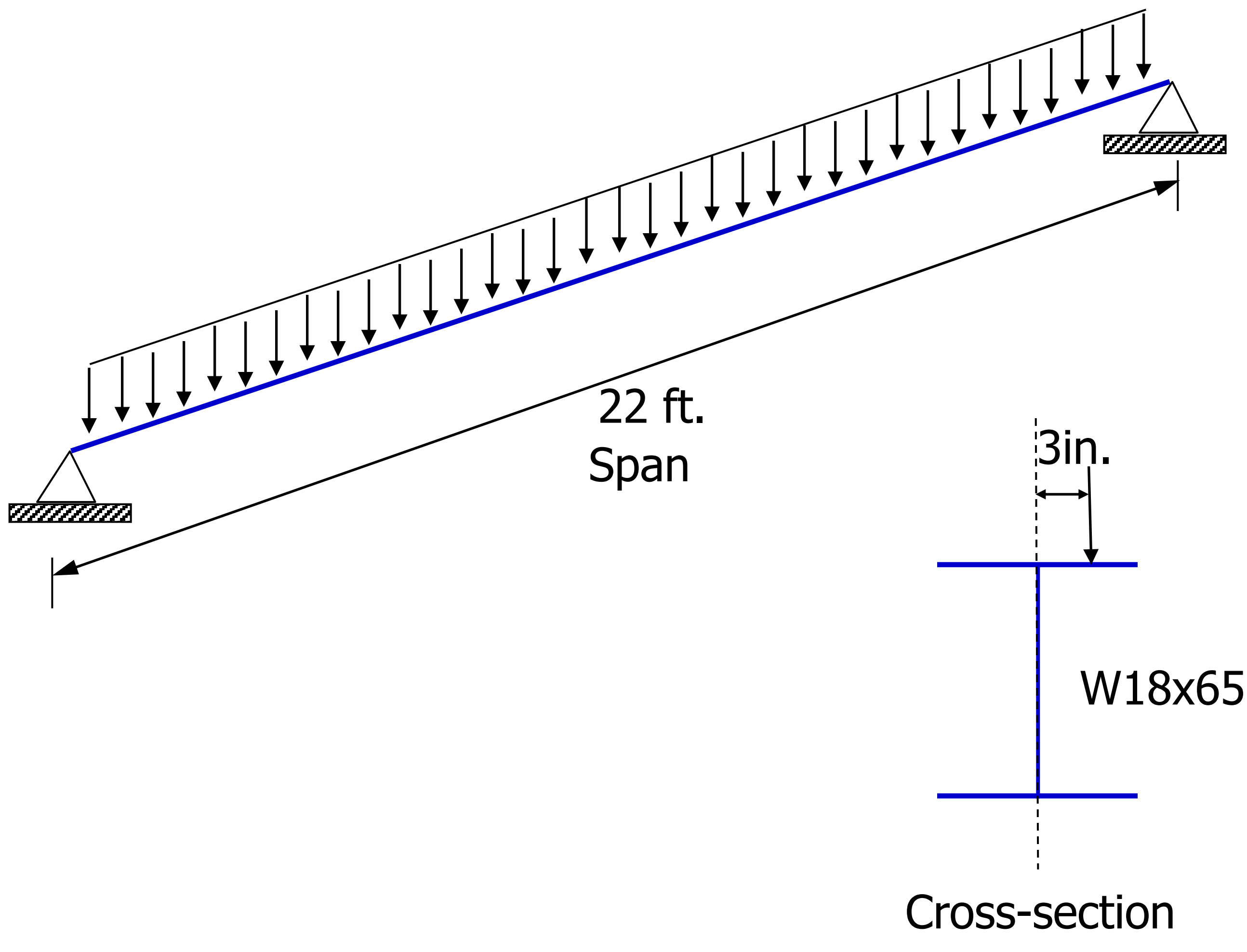
(4) Assumes – no influence of one force on other deformations

(5) Equations of equilibrium in the undeformed state.

HOMEWORK # 3

- Consider the 22 ft. long simply-supported W18x65 wide flange beam shown in Figure 1 below. It is subjected to a uniformly distributed load of 1k/ft that is placed with an eccentricity of 3 in. with respect to the centroid (and shear center).
- At the mid-span and the end support cross-sections, calculate the magnitude and distribution of:
  - Normal and shear stresses due to bending
  - Shear stresses due to pure torsion
  - Warping normal and shear stresses over the cross-section.
- Provide sketches and tables of the individual normal and shear stress distributions for each case.
- Superimpose the bending and torsional stress-states to determine the magnitude and location of maximum stresses.

HOMEWORK # 2



Chapter 2. – Second-Order Differential Equations

- This chapter focuses on deriving second-order differential equations governing the behavior of elastic members
- 2.1 – First order differential equations
- 2.2 – Second-order differential equations

2.2 Second-Order Differential Equations

- Governing the behavior of structural members
  - Elastic, Homogenous, and Isotropic
  - Strains and deformations are really small – small deflection theory
  - Equations of equilibrium in deformed state
  - The deformations and internal forces are no longer independent. They must be combined to consider effects.
- Consider the behavior of a member subjected to combined axial forces and bending moments at the ends. No torsional forces are applied explicitly – because that is very rare for CE structures.

Member model and loading conditions

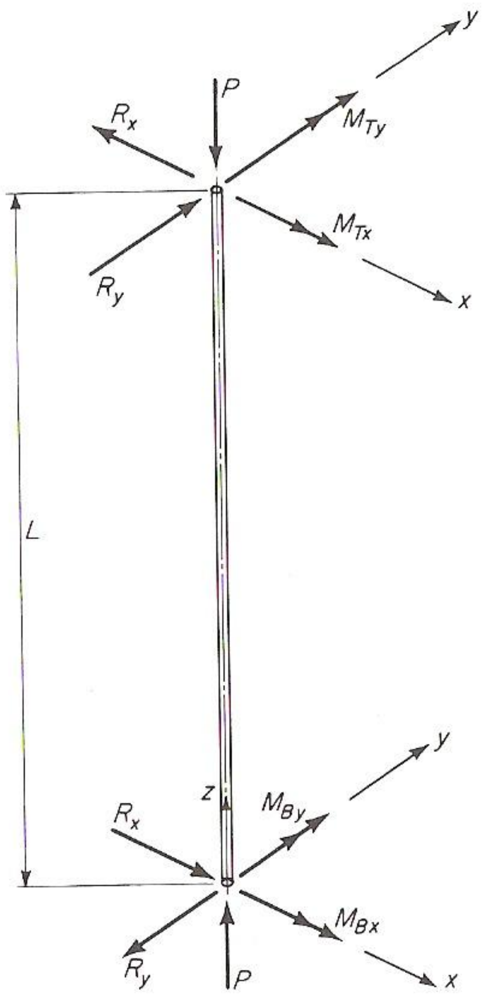


Fig. 2.30. End forces on a prismatic bar

- Member is initially straight and prismatic. It has a thin-walled open cross-section
- Member ends are pinned and prevented from translation.
- The forces are applied only at the member ends
- These consist only of axial and bending moment forces P, M_{TX}, M_{TY}, M_{BX}, M_{BY}
- Assume elastic behavior with small deflections
- Right-hand rule for positive moments and reactions and P assumed positive.

Member displacements (cross-sectional)

- Consider the middle line of thin-walled cross-section
- x and y are principal coordinates through centroid **C**
- Q is any point on the middle line. It has coordinates (x, y).
- Shear center **S** coordinates are (x_o, y_0)
- Shear center **S** displacements are *u*, *v*, and ϕ

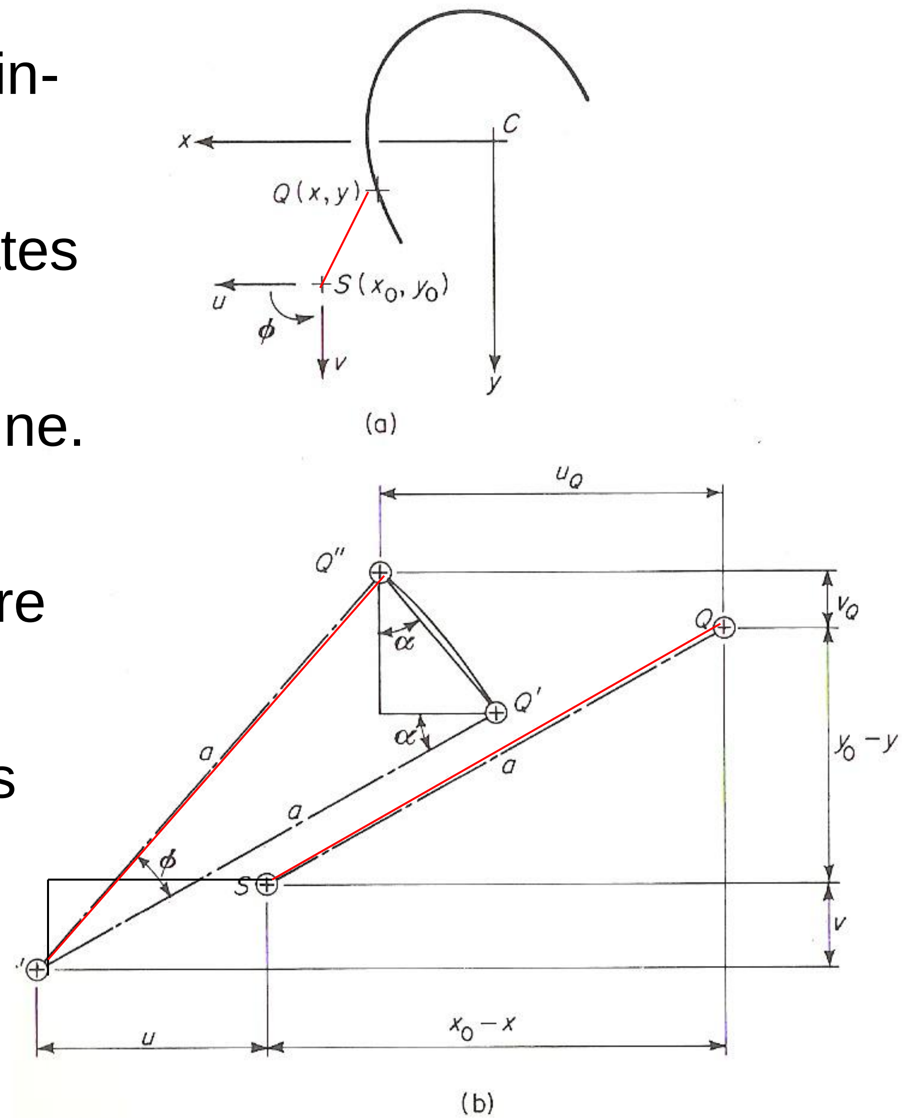


Fig. 2.31. Displacement of a point q in a cross section

Member displacements (cross-sectional)

- Displacements of Q are:
  $u_Q = u + a\,\phi \sin\alpha$
  $v_Q = v - a\,\phi\,\cos\alpha$
  where a is the distance from (
- But, $\sin\alpha = (y_0 - y) / a$
  $\cos\alpha = (x_0 - x) / a$
- Therefore, displacements of (
  $u_Q = u + \phi\,(y_0 - y)$
  $v_Q = v - \phi\,(x_0 - x)$
- Displacements of centroid C a
  $u_c = u + \phi\,(y_0)$
  $v_c = v - \phi\,(x_0)$

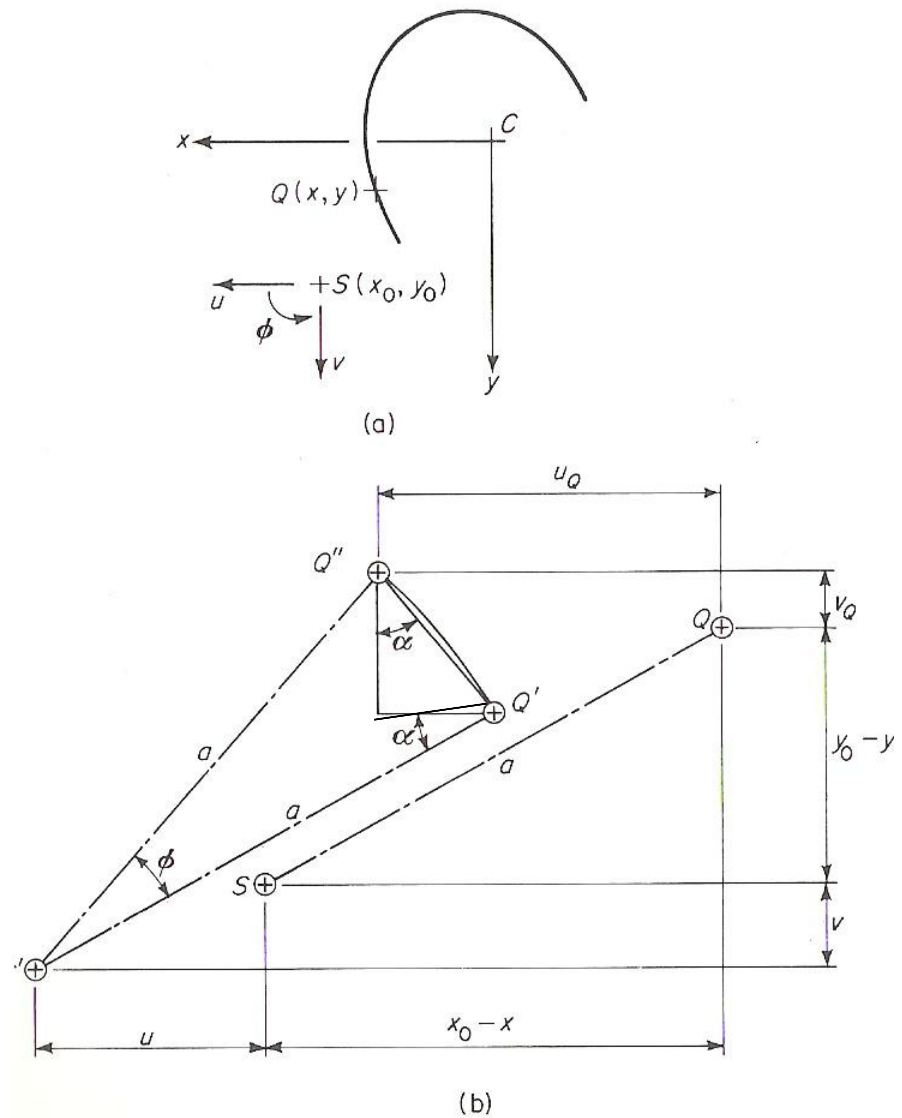


Fig. 2.31. Displacement of a point q in a cross section

Internal forces – second-order effects

- Consider the free body diagrams of the member in the deformed state.
- Look at the deformed state in the x-z and y-z planes in this Figure.
- The internal resisting moment at a distance z from the lower end are:

$$M_x = - M_{BX} + R_y\, z + P\, v_c$$

$$M_y = - M_{BY} + R_x\, z - P\, u_c$$

- The end reactions R_x and R_y are:

$$R_x = (M_{TY} + M_{BY}) / L$$

$$R_y = (M_{TX} + M_{BX}) / L$$

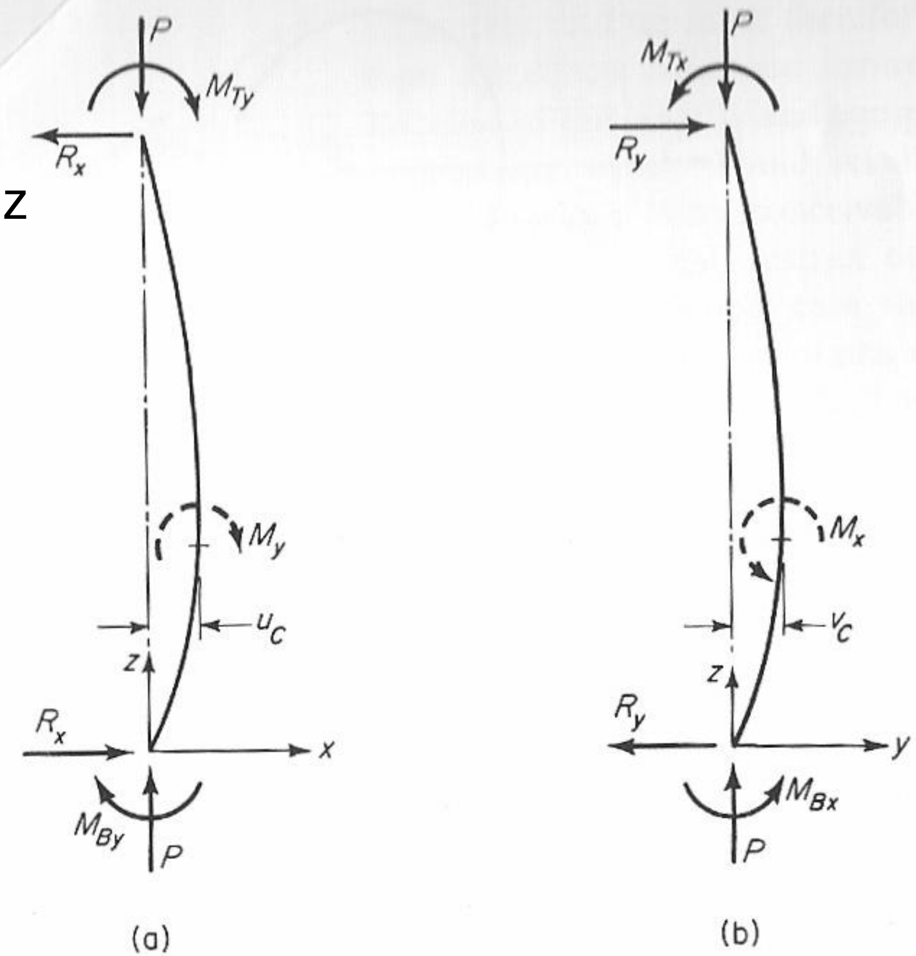


Fig. 2.32. Forces in the x-z and the y-z plane

Internal forces – second-order effects

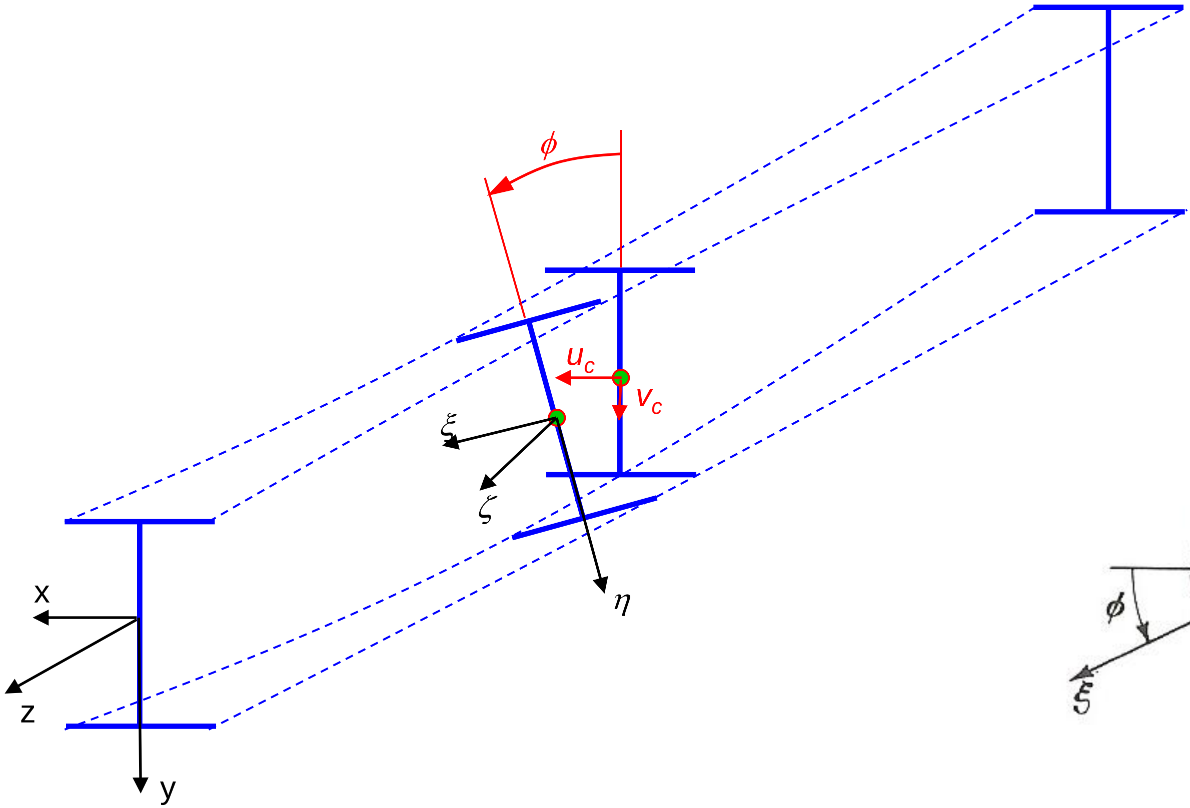
- Therefore,

$$M_x = -M_{BX} + \frac{z}{L}\left(M_{TX} + M_{BX}\right) + P\left(v - \phi\, x_0\right)$$

$$M_y = -M_{BY} + \frac{z}{L}\left(M_{TY} + M_{BY}\right) - P\left(u + \phi\, y_0\right)$$

Internal forces in the deformed state

- In the deformed state, the cross-section is such that the principal coordinate systems are changed from x-y-z to the $\xi-\eta-\zeta$ system



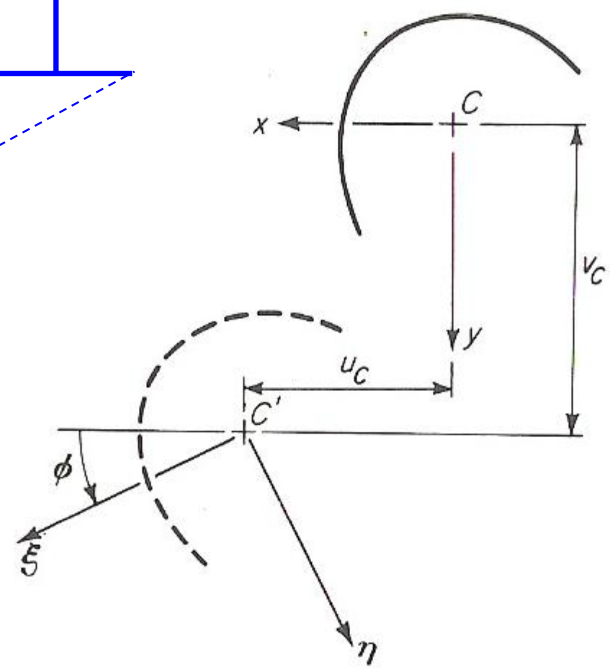
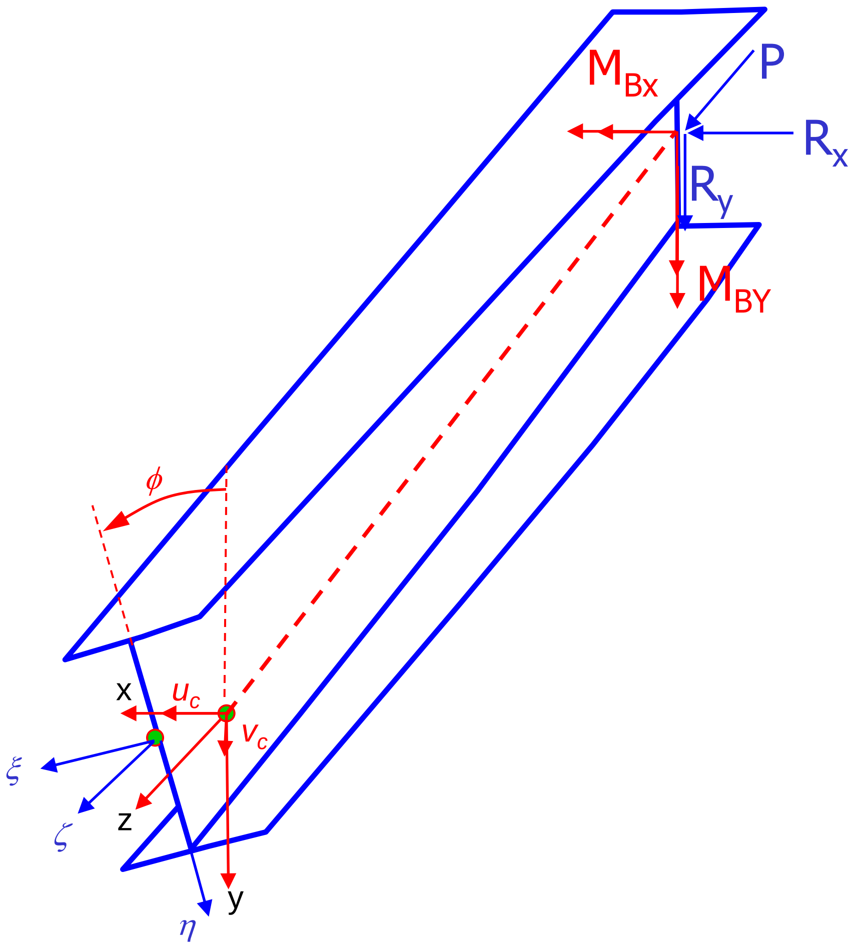
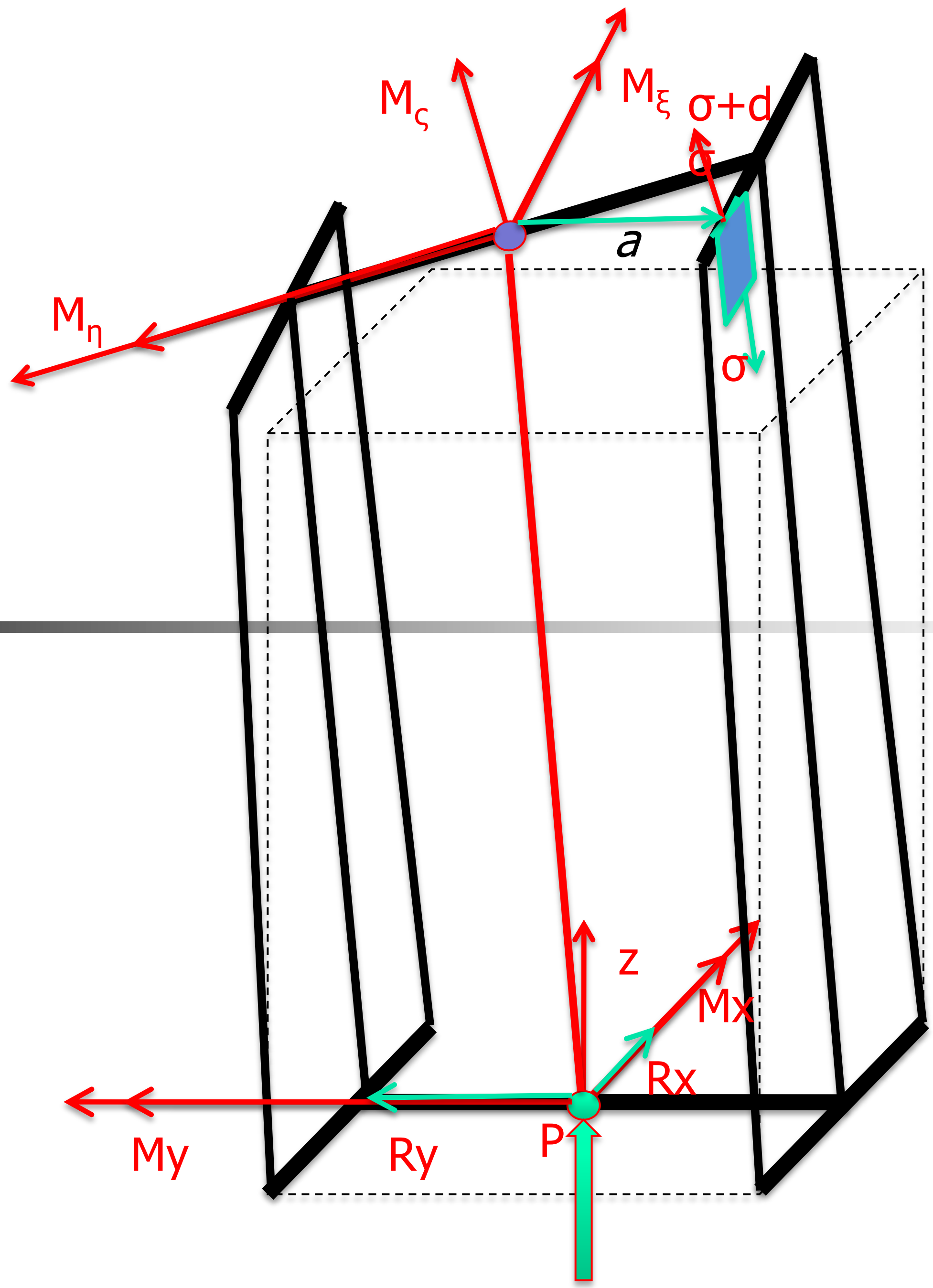


Fig. 2.33. Definition of the ξ-η coordinate system



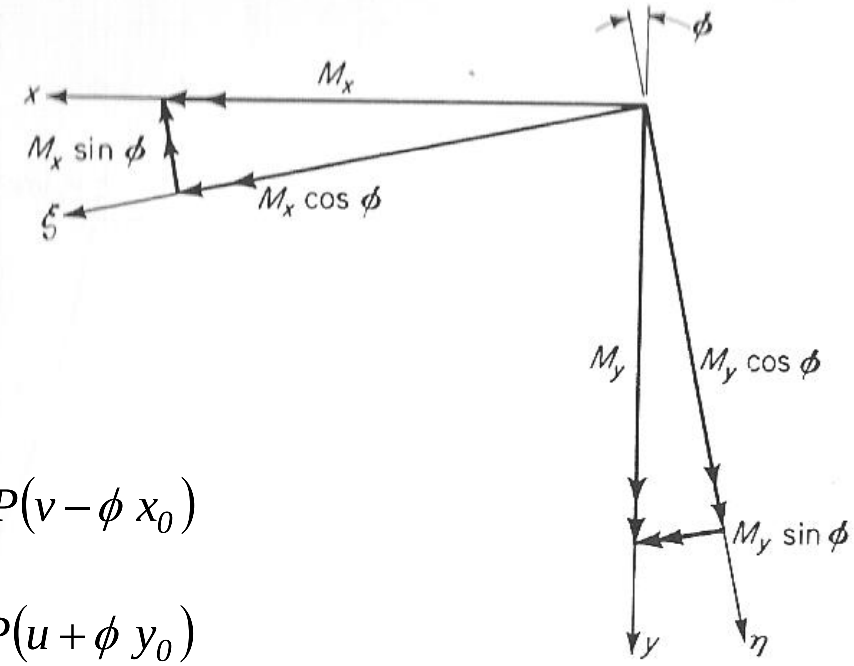
M_Bx
P
R_x
R_y
M_BY
φ
x
u_c
v_c
ξ
ζ
z
y
η



Mς
Mξ
σ+d
σ
Mη
a
σ
z
Mx
Rx
My
Ry
P

Internal forces in the deformed state

- The internal forces M_x and M_y must be transformed to these new ξ–η–ζ axes
- Since the angle ϕ is small
- $M_\xi = M_x + \phi M_y$
- $M_\eta = M_y - \phi M_x$



$$M_x = -M_{BX} + \frac{z}{L}\left(M_{TX} + M_{BX}\right) + P\left(v - \phi\, x_0\right)$$

$$M_y = -M_{BY} + \frac{z}{L}\left(M_{TY} + M_{BY}\right) - P\left(u + \phi\, y_0\right)$$

$$\therefore M_\xi = -M_{BX} + \frac{z}{L}\left(M_{TX} + M_{BX}\right) + P\,v - \phi\left(P\,x_0 + M_{BY} - \frac{z}{L}\left(M_{TY} + M_{BY}\right)\right)$$

$$\therefore M_\eta = -M_{BY} + \frac{z}{L}\left(M_{TY} + M_{BY}\right) + P\,u + \phi\left(-P\,y_0 + M_{BX} - \frac{z}{L}\left(M_{TX} + M_{BX}\right)\right)$$

Twisting component of internal forces

- Twisting moments M_ζ are produced by the internal and external forces
- There are four components contributing to the total M_ζ

(1) Contribution from M_x and M_y – $M_{\zeta 1}$

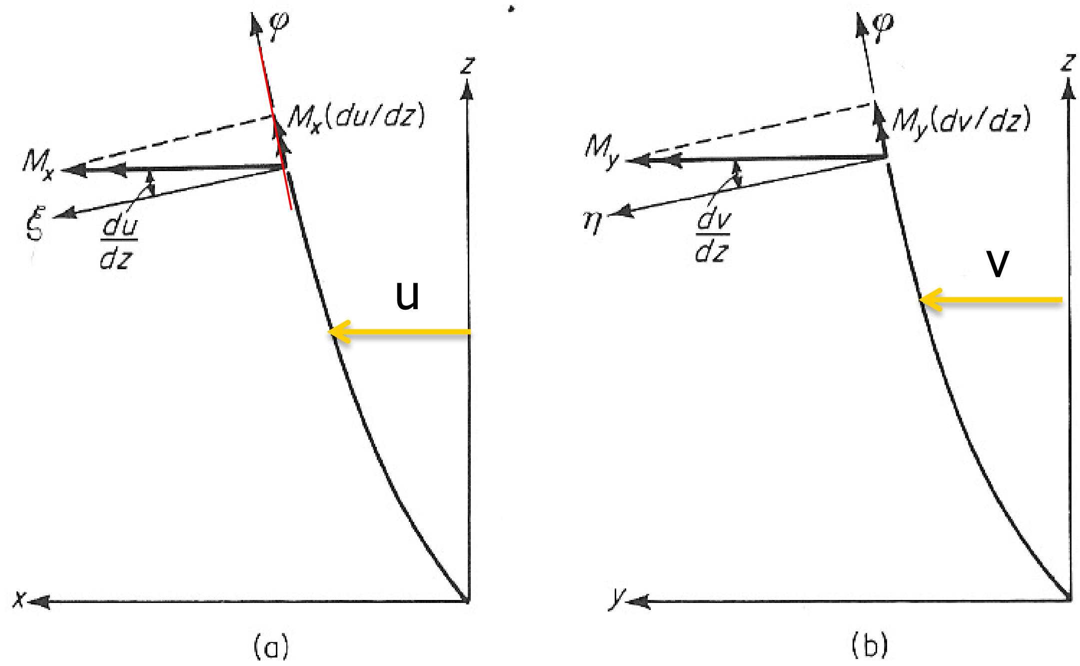
(2) Contribution from axial force P – $M_{\zeta 2}$

(3) Contribution from normal stress σ – $M_{\zeta 3}$

(4) Contribution from end reactions R_x and R_y – $M_{\zeta 4}$

- The total twisting moment $M_\zeta = M_{\zeta 1} + M_{\zeta 2} + M_{\zeta 3} + M_{\zeta 4}$

Twisting component – 1 of 4



- Twisting moment due to M_x & M_y
- $M_{\zeta 1} = M_x \sin(du/dz) + M_y \sin(dv/dz)$
- Therefore, due to small angles, $M_{\zeta 1} = M_x\, du/dz + M_y\, dv/dz$
- $M_{\zeta 1} = M_x u' + M_y v'$

Twisting component – 2 of 4

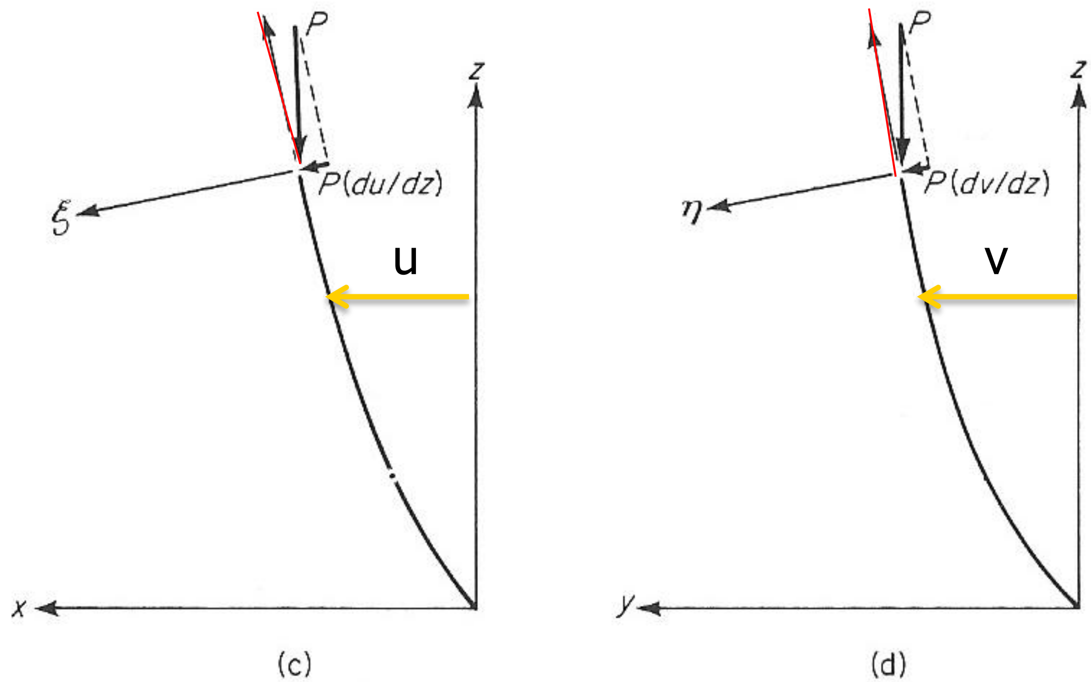


Fig. 2.35. Twisting due to components of M_x, M_y, and P

- The axial load P acts along the original vertical direction
- In the deformed state of the member, the longitudinal axis ζ is not vertical. Hence P will have components producing shears.
- These components will act at the centroid where P acts and will have values as shown above – assuming small angles

Twisting component – 2 of 4

- These shears will act at the centroid **C**, which is eccentric with respect to the shear center **S**. Therefore, they will produce secondary twisting.

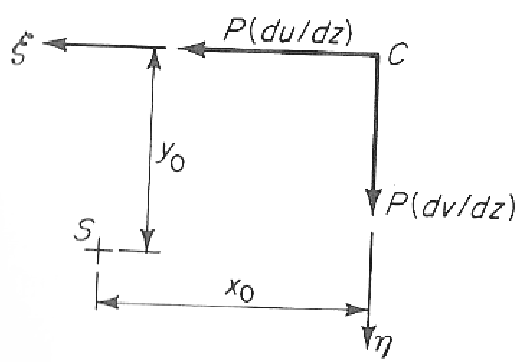


Fig. 2.36. Twisting due to the components of *P*

- $M_{\zeta 2} = P\,(y_0\, du/dz - x_0\, dv/dz)$
- Therefore, $M_{\zeta 2} = P\,(y_0\, u' - x_0\, v')$

Twisting component – 3 of 4

- The end reactions (shears) R_x and R_y act at the shear center **S** at the ends. But, along the member ends, the shear center will move by *u*, *v*, and ϕ.
- Hence, these reactions will also have a twisting effect produced by their eccentricity with respect to the shear center **S**.
- $M_{\zeta 4} + R_y u + R_x v = 0$
- Therefore,
- $M_{\zeta 4} = -(M_{TY} + M_{BY}) v/L - (M_{TX} + M_{BX}) u/L$

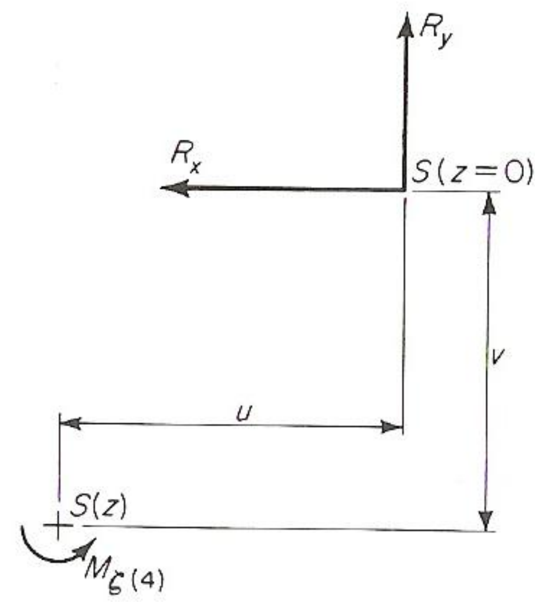


Fig. 2.38. Twisting due to the end shears

Twisting component – 4 of 4

- Wagner's effect or contribution – complicated.
- Two cross-sections that are $d\zeta$ apart will warp with respect to each other.
- The stress element $\sigma\, dA$ will become inclined by angle ($a\, d\phi/d\zeta$) with respect to $d\zeta$ axis.
- Twist produced by each stress element about **S** is equal to

$$dM_{\zeta 3} = -a(\sigma\, dA)\left(a\,\frac{d\phi}{d\zeta}\right)$$

$$\therefore M_{\zeta 3} = -\frac{d\phi}{d\zeta}\int_A \sigma\, a^2 dA$$

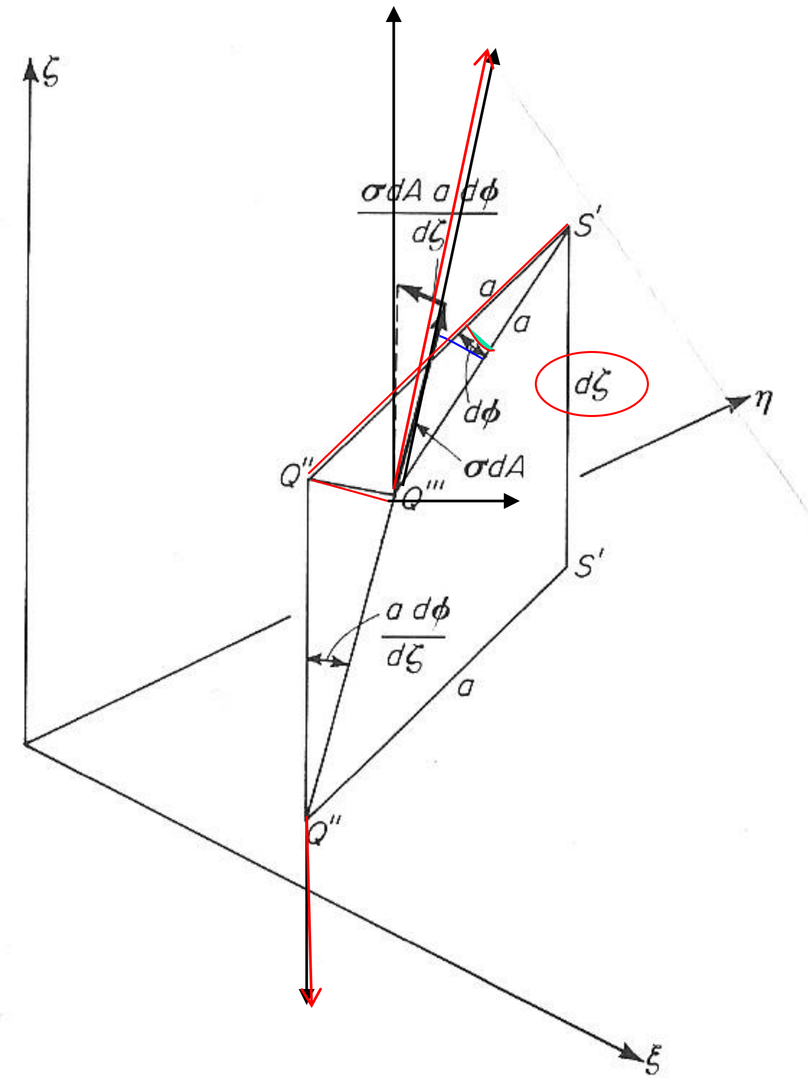


Fig. 2.37. Twisting due to the differential warping of two adjacent cross sections

Twisting component – 4 of 4

$$Let, \int_A \sigma\, a^2 dA = \overline{K}$$

$$\therefore M_{\zeta 3} = -\overline{K}\frac{d\phi}{d\zeta}$$

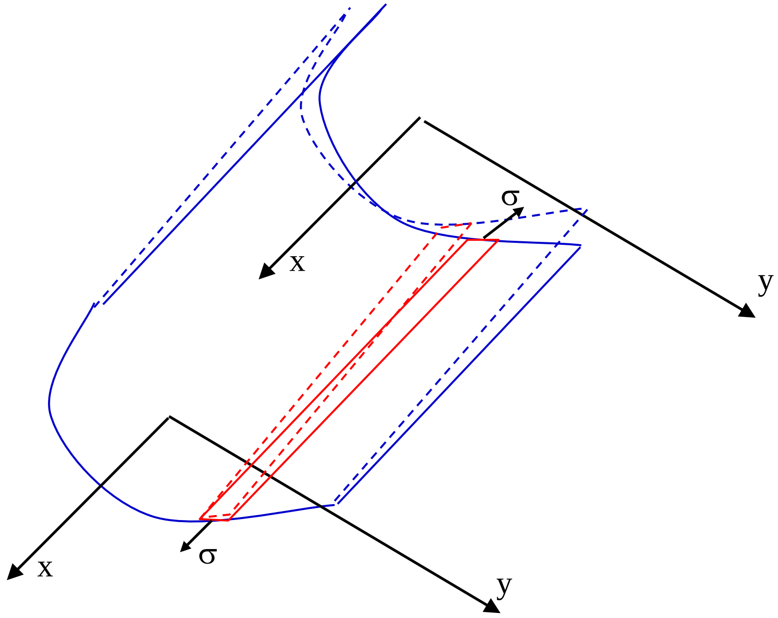
$$\therefore M_{\zeta 3} = -\overline{K}\frac{d\phi}{dz} \quad \cdots\cdots \text{ for small angles}$$

Twisting component – 4 of 4

$$Let, \int_A \sigma\, a^2 dA = \bar{K}$$

$$\therefore M_{\zeta 3} = -\bar{K} \frac{d\phi}{d\zeta}$$

$$\therefore M_{\zeta 3} = -\bar{K} \frac{d\phi}{dz} \quad \cdots\cdots \text{ for small angles}$$



Total Twisting Component

- $M_{\zeta} = M_{\zeta 1} + M_{\zeta 2} + M_{\zeta 3} + M_{\zeta 4}$

  $M_{\zeta 1} = M_x u' + M_y v'$

  $M_{\zeta 2} = P (y_0 u' - x_0 v')$

  $M_{\zeta 4} = -(M_{TY} + M_{BY}) v/L - (M_{TX} + M_{BX}) u/L$

  $M_{\zeta 3} = -\underline{K} \phi'$

- Therefore,

$$M_{\zeta} = M_x u' + M_y v' + P (y_0 u' - x_0 v') - (M_{TY} + M_{BY}) v/L - (M_{TX} + M_{BX}) u/L - \underline{K} \phi'$$

- While,

$$M_{\xi} = -M_{BX} + \frac{z}{L}(M_{TX} + M_{BX}) + P v - \phi \left(P x_0 + M_{BY} - \frac{z}{L}(M_{TY} + M_{BY}) \right)$$

$$M_{\eta} = -M_{BY} + \frac{z}{L}(M_{TY} + M_{BY}) + P u + \phi \left(-P y_0 + M_{BX} - \frac{z}{L}(M_{TX} + M_{BX}) \right)$$

Total Twisting Component

- $M_{\zeta} = M_{\zeta 1} + M_{\zeta 2} + M_{\zeta 3} + M_{\zeta 4}$

  $M_{\zeta 1} = M_x u' + M_y v'$ $\quad$ $M_{\zeta 2} = P (y_0 u' - x_0 v')$ $\quad$ $M_{\zeta 3} = -\underline{K} \phi'$

  $M_{\zeta 4} = -(M_{TY} + M_{BY}) v/L - (M_{TX} + M_{BX}) u/L$

- Therefore,

$$\therefore M_{\zeta} = M_x u' + M_y v' + P(y_0 u' - x_0 v') - (M_{TY} + M_{BY})\frac{v}{L} - (M_{TX} + M_{BX})\frac{u}{L} - \bar{K}\phi'$$

$$\therefore M_{\zeta} = (M_x + P y_0) u' + (M_y - P x_0) v' - (M_{TY} + M_{BY})\frac{v}{L} - (M_{TX} + M_{BX})\frac{u}{L} - \bar{K}\phi'$$

$$But, M_x = -M_{BX} + \frac{z}{L}(M_{BX} + M_{TX}) + P(v - \phi x_0)$$

$$and, M_y = -M_{BY} + \frac{z}{L}(M_{BY} + M_{TY}) - P(u + \phi y_0)$$

$$\therefore M_{\zeta} = (-M_{BX} - \frac{z}{L}(M_{BX} + M_{TX}) + P y_0) u' + (-M_{BY} - \frac{z}{L}(M_{BY} + M_{TY}) - P x_0) v'$$

$$-(M_{TY} + M_{BY})\frac{v}{L} - (M_{TX} + M_{BX})\frac{u}{L} - \bar{K}\phi'$$

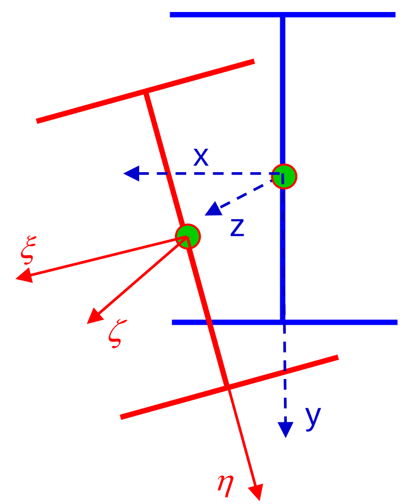
Internal moments about the $\xi-\eta-\zeta$ axes

- Thus, now we have the internal moments about the $\xi-\eta-\zeta$ axes for the deformed member cross-section.

$$M_\xi = -M_{BX} + \frac{z}{L}\left(M_{TX} + M_{BX}\right) + P\,v - \phi\left(P\,x_0 + M_{BY} - \frac{z}{L}\left(M_{TY} + M_{BY}\right)\right)$$

$$M_\eta = -M_{BY} + \frac{z}{L}\left(M_{TY} + M_{BY}\right) - P\,u + \phi\left(-P\,y_0 + M_{BX} - \frac{z}{L}\left(M_{TX} + M_{BX}\right)\right)$$

$$M_\zeta = \left(-M_{BX} - \frac{z}{L}\left(M_{BX} + M_{TX}\right) + P\,y_0\right)u' + \left(-M_{BY} - \frac{z}{L}\left(M_{BY} + M_{TY}\right) - P\,x_0\right)v'$$
$$-\left(M_{TY} + M_{BY}\right)\frac{v}{L} - \left(M_{TX} + M_{BX}\right)\frac{u}{L} - \bar{K}\,\phi'$$



Internal Moment – Deformation Relations

- The internal moments M_ξ, M_η, *and* M_ζ will still produce flexural bending about the centroidal principal axis and twisting about the shear center.
- The flexural bending about the principal axes will produce linearly varying longitudinal stresses.
- The torsional moment will produce longitudinal and shear stresses due to warping and pure torsion.
- The differential equations relating moments to deformations are still valid. Therefore,

$$M_\xi = - E\, I_\xi\, v'' \ldots\ldots\ldots\ldots\ldots\ldots\ldots (I_\xi = I_x)$$

$$M_\eta = E\, I_\eta\, u'' \ldots\ldots\ldots\ldots\ldots\ldots\ldots (I_\eta = I_y)$$

$$M_\zeta = G\, K_T\, \phi' - E\, I_w\, \phi'''$$

Internal Moment – Deformation Relations

Therefore,

$$M_{\xi} = -E\, I_x\, v'' = -M_{BX} + \frac{z}{L}\left(M_{TX} + M_{BX}\right) + P\, v - \phi\left(P\, x_0 + M_{BY} - \frac{z}{L}\left(M_{TY} + M_{BY}\right)\right)$$

$$M_{\eta} = E\, I_y\, u'' = -M_{BY} + \frac{z}{L}\left(M_{TY} + M_{BY}\right) - P\, u + \phi\left(-P\, y_0 + M_{BX} - \frac{z}{L}\left(M_{TX} + M_{BX}\right)\right)$$

$$M_{\zeta} = G\, K_T\, \phi' - E\, I_w\, \phi''' = \left(-M_{BX} - \frac{z}{L}(M_{BX} + M_{TX}) + P\, y_0\right) u' +$$

$$\left(-M_{BY} - \frac{z}{L}(M_{BY} + M_{TY}) - P\, x_0\right) v' - (M_{TY} + M_{BY})\frac{v}{L} - (M_{TX} + M_{BX})\frac{u}{L} - \bar{K}\, \phi'$$

Second-Order Differential Equations

You end up with three coupled differential equations that relate the applied forces and moments to the deformations u, v, and ϕ.

Therefore,

1 $$E\,I_x\,v'' + P\,v - \phi\left(P\,x_0 + M_{BY} - \frac{z}{L}(M_{TY} + M_{BY})\right) = M_{BX} - \frac{z}{L}(M_{TX} + M_{BX})$$

2 $$E\,I_y\,u'' + P\,u - \phi\left(-P\,y_0 + M_{BX} - \frac{z}{L}(M_{TX} + M_{BX})\right) = -M_{BY} + \frac{z}{L}(M_{TY} + M_{BY})$$

3 $$E\,I_w\,\phi''' - (G\,K_T + \bar{K})\,\phi' + u'\left(-M_{BX} - \frac{z}{L}(M_{BX} + M_{TX}) + P\,y_0\right)$$

$$-v'\left(M_{BY} + \frac{z}{L}(M_{BY} + M_{TY}) + P\,x_0\right) - \frac{v}{L}(M_{TY} + M_{BY}) - \frac{u}{L}(M_{TX} + M_{BX}) = 0$$

These differential equations can be used to investigate the elastic behavior and buckling of beams, columns, beam-columns and also complete frames – that will form a major part of this course.

Chapter 3. Structural Columns

- 3.1 Elastic Buckling of Columns
- 3.2 Elastic Buckling of Column Systems – Frames
- 3.3 Inelastic Buckling of Columns
- 3.4 Column Design Provisions (U.S. and Abroad)

3.1 Elastic Buckling of Columns

- Start out with the second-order differential equations derived in Chapter 2. Substitute P=P and $M_{TY} = M_{BY} = M_{TX} = M_{BX} = 0$
- Therefore, the second-order differential equations simplify to:

1 $$E\,I_x\,v'' + P\,v - \phi\left(P\,x_0\right) = 0$$

2 $$E\,I_y\,u'' + P\,u - \phi\left(-P\,y_0\right) = 0$$

3 $$E\,I_w\,\phi''' - (G\,K_T + \bar{K})\,\phi' + u'\,(P\,y_0) - v'\,(P\,x_0) = 0$$

- This is all great, but before we proceed any further we need to deal with Wagner's effect – which is a little complicated.

Wagner's effect for columns

$$\bar{K}\,\phi' = \int_A \sigma\, a^2\, \phi' dA$$

where,

$$\sigma = -\frac{P}{A} + \frac{M_\xi\, y}{I_x} - \frac{M_\eta\, x}{I_y} + E\, W_n\, \phi''$$

$$M_\xi = P\,(v - \phi\, x_0)$$

$$M_\eta = -P\,(u + \phi\, y_0)$$

$$\therefore \bar{K}\,\phi' = \int_A \left[-\frac{P}{A} + \frac{P\,(v-\phi\, x_0)\, y}{I_x} - \frac{-P\,(u+\phi\, y_0)\, x}{I_y} + E\, W_n\, \phi'' \right] \phi'\, a^2\, dA$$

$$\therefore \bar{K}\,\phi' = \left[-\frac{P}{A} + \frac{P\,(v-\phi\, x_0)\, y}{I_x} - \frac{-P\,(u+\phi\, y_0)\, x}{I_y} + E\, W_n\, \phi'' \right] \phi' \int_A a^2\, dA$$

Neglecting higher order terms; $\bar{K}\,\phi' = -\frac{P}{A}\phi' \int_A a^2\, dA$

Wagner's effect for columns

$$But,\ a^2 = (x_0 - x)^2 + (y_0 - y)^2$$

$$\therefore \int_A a^2\ dA = \int_A (x_0 - x)^2 + (y_0 - y)^2\ dA$$

$$\therefore \int_A a^2\ dA = \int_A \left[x_0^2 + y_0^2 + x^2 + y^2 - 2\,x_0\,x - 2\,y_0\,y \right] dA$$

$$\therefore \int_A a^2\ dA = \left[x_0^2 + y_0^2 \right] \int_A dA + \int_A x^2 dA + \int_A y^2 dA - 2x_0 \int_A x\ dA - 2y_0 \int_A y\ dA$$

$$\therefore \int_A a^2\ dA = (x_0^2 + y_0^2)\ A + I_x + I_y$$

$$Finally,$$

$$\therefore \bar{K}\ \phi' = -\frac{P}{A} \left[(x_0^2 + y_0^2)\ A + I_x + I_y \right] \phi'$$

$$\therefore \bar{K}\ \phi' = -P \left[(x_0^2 + y_0^2) + \frac{I_x + I_y}{A} \right] \phi'$$

$$Let\ \bar{r}_0^{\,2} = \left[(x_0^2 + y_0^2) + \frac{I_x + I_y}{A} \right]$$

$$\therefore \bar{K}\ \phi' = -P\ \bar{r}_0^{\,2}\ \phi'$$

Second-order differential equations for columns

- Simplify to:

1 $$E\,I_x\,v'' + P\,v - \phi\left(P\,x_0\right) = 0$$

2 $$E\,I_y\,u'' + P\,u + \phi\left(P\,y_0\right) = 0$$

3 $$E\,I_w\,\phi''' + (P\,\bar{r}_0^{\,2} - G\,K_T)\,\phi' + u'\,(P\,y_0) - v'\,(P\,x_0) = 0$$

- Where

$$\bar{r}_0^{\,2} = x_0^2 + y_0^2 + \frac{I_x + I_y}{A}$$

Column buckling – doubly symmetric section

- For a doubly symmetric section, the shear center is located at the centroid $x_o = y_0 = 0$. Therefore, the three equations become uncoupled

1 $$E\,I_x\,v'' + P\,v = 0$$

2 $$E\,I_y\,u'' + P\,u = 0$$

3 $$E\,I_w\,\phi''' + (P\,\bar{r}_0^{\,2} - G\,K_T)\,\phi' = 0$$

- Take two derivatives of the first two equations and one more derivative of the third equation.

1 $$E\,I_x\,v^{iv} + P\,v'' = 0$$

2 $$E\,I_y\,u^{iv} + P\,u'' = 0$$

3 $$E\,I_w\,\phi^{iv} + (P\,\bar{r}_0^{\,2} - G\,K_T)\,\phi'' = 0$$

$$Let,\ F_v^2 = \frac{P}{E\,I_x} \qquad F_u^2 = \frac{P}{E\,I_y} \qquad F_\phi^2 = \frac{P\,\bar{r}_0^{\,2} - G\,K_T}{E\,I_w}$$

Column buckling – doubly symmetric section

1 $$v^{iv} + F_v^2\, v'' = 0$$

2 $$u^{iv} + F_u^2\, u'' = 0$$

3 $$\phi^{iv} + F_\phi^2 \phi'' = 0$$

- All three equations are similar and of the fourth order. The solution will be of the form $C_1 \sin \lambda z + C_2 \cos \lambda z + C_3 z + C_4$
- Need four boundary conditions to evaluate the constant $C_1..C_4$
- For the simply supported case, the boundary conditions are:

  $u = u'' = 0;\ v = v'' = 0;\ \phi = \phi'' = 0$

- Lets solve one differential equation – the solution will be valid for all three.

Column buckling – doubly symmetric section

$$v^{iv} + F_v^2\, v'' = 0$$

Solution is

$$v = C_1 \sin F_v z + C_2 \cos F_v z + C_3 z + C_4$$

$$\therefore v'' = -C_1\, F_v^2 \sin F_v z - C_2\, F_v^2 \cos F_v z$$

Boundary conditions :

$$v(0) = v''(0) = v(L) = v''(L) = 0$$

$$C_2 + C_4 = 0 \qquad \cdots\cdots v(0) = 0$$

$$C_2 = 0 \qquad \cdots\cdots v''(0) = 0$$

$$C_1 \sin F_v L + C_2 \cos F_v L + C_3 L + C_4 \qquad \cdots\cdots v(L) = 0$$

$$-C_1\, F_v^2 \sin F_v L - C_2\, F_v^2 \cos F_v L \qquad \cdots\cdots v''(L) = 0$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \sin F_v L & \cos F_v L & L & 1 \\ -F_v^2 \sin F_v L & -F_v^2 \cos F_v L & 0 & 0 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$The\ \left| coefficient\ matrix \right| = 0$$

$$\therefore F_v^2 \sin F_v L = 0$$

$$\therefore \sin F_v L = 0$$

$$\therefore F_v L = n\,\pi$$

$$\therefore F_v = \sqrt{\frac{P}{E\, I_x}} = \frac{n\,\pi}{L}$$

$$\therefore P_x = \frac{n^2\,\pi^2}{L^2} E\, I_x$$

Smallest value of $n = 1$:

$$\therefore \boxed{P_x = \frac{\pi^2\, E\, I_x}{L^2}}$$

Column buckling – doubly symmetric section

$$\textit{Similarly,}$$
$$\sin F_u L = 0$$
$$\therefore F_u L = n\,\pi$$
$$\therefore F_u = \sqrt{\frac{P}{E\,I_y}} = \frac{n\,\pi}{L}$$
$$\therefore P_y = \frac{n^2\,\pi^2}{L^2} E\,I_y$$
$$\textit{Smallest value of } n = 1: \quad \boxed{P_y = \frac{\pi^2\,E\,I_y}{L^2}}$$

$$\textit{Similarly,}$$
$$\sin F_\phi L = 0$$
$$\therefore F_\phi L = n\,\pi$$
$$\therefore F_\phi = \sqrt{\frac{P\,\overline{r}_0^2 - G\,K_T}{E\,I_w}} = \frac{n\,\pi}{L}$$
$$\therefore P_\phi = \left(\frac{n^2\,\pi^2}{L^2} E\,I_w + G\,K_T\right)\frac{1}{\overline{r}_0^2}$$
$$\textit{Smallest value of } n = 1:$$
$$\boxed{P_\phi = \left(\frac{n^2\,\pi^2}{L^2} E\,I_w + G\,K_T\right)\frac{1}{\overline{r}_0^2}}$$

Summary

$$P_x = \frac{\pi^2\,E\,I_x}{L^2} \qquad 1$$
$$P_y = \frac{\pi^2\,E\,I_y}{L^2} \qquad 2$$
$$P_\phi = \left[\frac{\pi^2\,E\,I_w}{L^2} + G\,K_T\right]\frac{1}{\overline{r}_0^2} \qquad 3$$

Column buckling – doubly symmetric section

- Thus, for a doubly symmetric cross-section, there are three distinct buckling loads P_x, P_y, and P_z.
- The corresponding buckling modes are:

  $v = C_1 \sin(\pi z/L)$, $u = C_2 \sin(\pi z/L)$, and $\phi = C_3 \sin(\pi z/L)$.

- These are, flexural buckling about the x and y axes and torsional buckling about the z axis.
- As you can see, the three buckling modes are uncoupled. You must compute all three buckling load values.
- The smallest of three buckling loads will govern the buckling of the column.

Column buckling – boundary conditions

Consider the case of fix-fix boundary conditions:

$$v^{iv} + F_v^2\, v'' = 0$$

Solution is

$$v = C_1 \sin F_v z + C_2 \cos F_v z + C_3 z + C_4$$

$$\therefore v' = C_1\, F_v \cos F_v z - C_2\, F_v \sin F_v z + C_3$$

Boundary conditions:

$$v(0) = v'(0) = v(L) = v'(L) = 0$$

$$\therefore C_2 + C_4 = 0 \qquad \cdots v(0) = 0$$

$$C_1\, F_v + C_3 = 0 \qquad \cdots v'(0) = 0$$

$$C_1 \sin F_v L + C_2 \cos F_v L + C_3 L + C_4 \qquad \cdots v(L) = 0$$

$$C_1\, F_v \cos F_v L - C_2\, F_v \sin F_v L + C_3 \qquad \cdots v'(L) = 0$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ F_v & 0 & 1 & 0 \\ \sin F_v L & \cos F_v L & L & 1 \\ F_v\, \cos F_v L & -F_v\, \sin F_v L & 1 & 0 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The $\left| coefficient\ matrix \right| = 0$

$$\therefore F_v\, L \sin F_v L - 2\cos F_v L + 2 = 0$$

$$\therefore 2\sin \frac{F_v L}{2} \left[F_v L \cos \frac{F_v L}{2} + 2\sin \frac{F_v L}{2} \right] = 0$$

$$\therefore \frac{F_v L}{2} = n\,\pi$$

$$\therefore F_v = \frac{2\,n\,\pi}{L}$$

$$\therefore P_x = \frac{4n^2\,\pi^2}{L^2} E\, I_x$$

Smallest value of $n = 1$:

$$\therefore \boxed{P_x = \frac{\pi^2\, E\, I_x}{(0.5\,L)^2} = \frac{\pi^2\, E\, I_x}{(K\,L)^2}}$$

Column Boundary Conditions

- The critical buckling loads for columns with different boundary conditions can be expressed as:

$$P_x = \frac{\pi^2 E I_x}{(K_x L)^2} \qquad 1$$

$$P_y = \frac{\pi^2 E I_y}{(K_y L)^2} \qquad 2$$

$$P_\phi = \left[\frac{\pi^2 E I_w}{(K_z L)^2} + G K_T\right] \frac{1}{\bar{r}_0^2} \qquad 3$$

- Where, K_x, K_y, and K_z are functions of the boundary conditions:
- K=1 for simply supported boundary conditions
- K=0.5 for fix-fix boundary conditions
- K=0.7 for fix-simple boundary conditions

Column buckling – example.

- Consider a wide flange column W27 x 84. The boundary conditions are:

  $v=v''=u=u'=\phi=\phi'=0$ at z=0, and $v=v''=u=u'=\phi=\phi''=0$ at z=L

- For flexural buckling about the x-axis – simply supported – K_x=1.0
- For flexural buckling about the y-axis – fixed at both ends – K_y = 0.5
- For torsional buckling about the z-axis – pin-fix at two ends - K_z=0.7

$$P_x = \frac{\pi^2 E I_x}{(K_x L)^2} = \frac{\pi^2 E A r_x^2}{(K_x L)^2} = \frac{\pi^2 E A}{\left(K_x \dfrac{L}{r_x}\right)^2}$$

$$P_y = \frac{\pi^2 E I_y}{(K_y L)^2} = \frac{\pi^2 E A r_y^2}{(K_y L)^2} = \frac{\pi^2 E A}{\left(K_y \dfrac{L}{r_x}\right)^2}\left(\frac{r_y}{r_x}\right)^2$$

$$P_\phi = \left[\frac{\pi^2 E I_w}{(K_z L)^2} + G K_T\right]\frac{1}{\bar{r}_0^2} = \left[\frac{\pi^2 E I_w}{\left(K_z \dfrac{L}{r_x}\right)^2} + G K_T r_x^2\right]\frac{A}{r_x^2 \times (I_x + I_y)}$$

Column buckling – example.

$$\therefore \frac{P_x}{P_Y} = \frac{\pi^2\, E\, A}{\left(K_x\, \frac{L}{r_x}\right)^2} \times \frac{1}{A\,\sigma_Y} = \frac{\pi^2\, E}{\sigma_Y \left(K_x\, \frac{L}{r_x}\right)^2} = \frac{5823.066}{\left(\frac{L}{r_x}\right)^2}$$

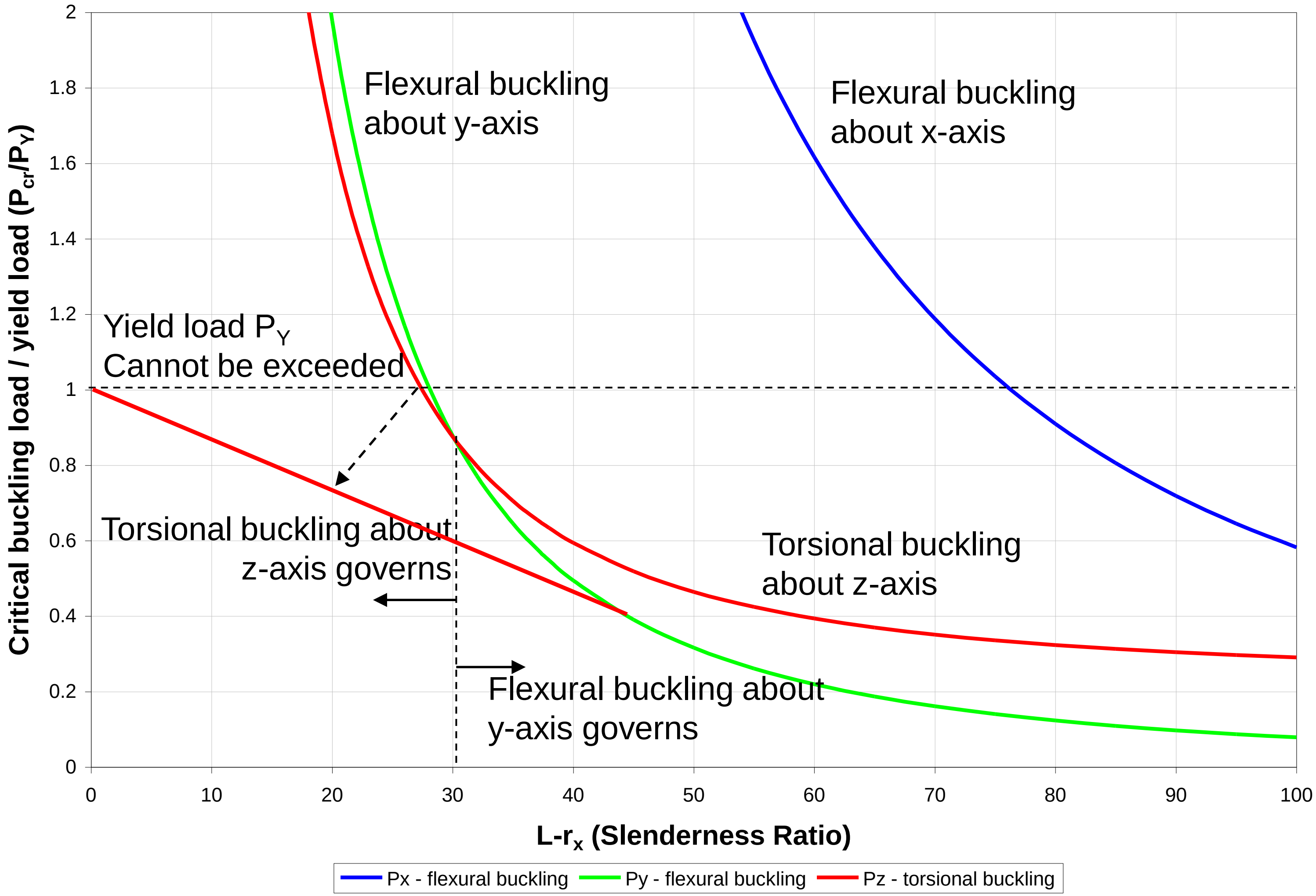
$$\frac{P_y}{P_Y} = \frac{\pi^2\, E\, A}{\left(K_y\, \frac{L}{r_x}\right)^2} \times \frac{(r_y / r_x)^2}{A\,\sigma_Y} = \frac{\pi^2\, E\, (r_y / r_x)^2}{\sigma_Y \left(K_y\, \frac{L}{r_x}\right)^2} = \frac{791.02}{\left(\frac{L}{r_x}\right)^2}$$

$$\frac{P_\phi}{P_Y} = \left[\frac{\pi^2\, E\, I_w}{\left(K_z\, \frac{L}{r_x}\right)^2} + G\, K_T\, r_x^2\right] \frac{A}{r_x^2 \times \left(I_x + I_y\right)} \times \frac{1}{A\,\sigma_Y}$$

$$\therefore \frac{P_\phi}{P_Y} = \left[\frac{\pi^2\, E\, I_w}{\left(K_z\, \frac{L}{r_x}\right)^2} + G\, K_T\, r_x^2\right] \frac{1}{r_x^2 \times \left(I_x + I_y\right) \times \sigma_Y}$$

$$\therefore \frac{P_\phi}{P_Y} = \frac{578.26}{\left(\frac{L}{r_x}\right)^2} + 0.2333$$

Column buckling – example.



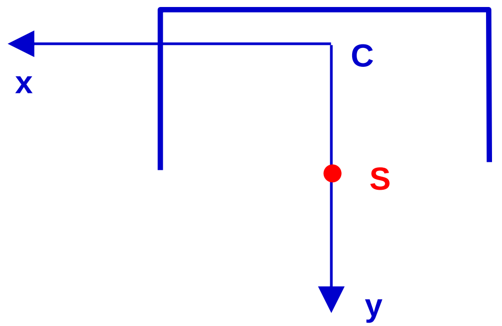
Column buckling – example.

- When L is such that $L/r_x < 31$; torsional buckling will govern
- $r_x = 10.69$ in. Therefore, $L/r_x = 31 \rightarrow L=338$ in.=28 ft.
- Typical column length =10 – 15 ft. Therefore, typical $L/r_x = 11.2 - 16.8$
- Therefore elastic torsional buckling will govern.
- But, the predicted load is much greater than P_Y. Therefore, inelastic buckling will govern.

- Summary – Typically must calculate all three buckling load values to determine which one governs. However, for common steel buildings made using wide flange sections – the minor (y-axis) flexural buckling usually governs.
- In this problem, the torsional buckling governed because the end conditions for minor axis flexural buckling were fixed. This is very rarely achieved in common building construction.

Column Buckling – Singly Symmetric Columns

- Well, what if the column has only one axis of symmetry. Like the x-axis or the y-axis or so.



- As shown in this figure, the y – axis is the axis of symmetry.
- The shear center S will be located on this axis.
- Therefore $x_0 = 0$.
- The differential equations will simplify to:

1 $$E\,I_x\,v'' + P\,v = 0$$

2 $$E\,I_y\,u'' + P\,u + \phi\left(P\,y_0\right) = 0$$

3 $$E\,I_w\,\phi''' + (P\,\overline{r}_0^{\,2} - G\,K_T)\,\phi' + u'\,(P\,y_0) = 0$$

Column Buckling – Singly Symmetric Columns

- The first equation for flexural buckling about the x-axis (axis of non-symmetry) becomes uncoupled.

$$E\,I_x\,v'' + P\,v = 0 \quad \cdots\cdots(1)$$

$$\therefore E\,I_x\,v^{iv} + P\,v'' = 0$$

$$\therefore v^{iv} + F_v^{\,2}\,v'' = 0$$

$$where,\ F_v^{\,2} = \frac{P}{E\,I_x}$$

$$\therefore v = C_1 \sin F_v z + C_2 \cos F_v z + C_3 z + C_4$$

Boundary conditions

$$\sin F_v L = 0$$

$$\therefore P_x = \frac{\pi^2\,E\,I_x}{(K_x\,L_x)^2}$$

$$Buckling\ \mathrm{mod}\ v = C_1 \sin F_v z$$

- Equations (2) and (3) are still coupled in terms of *u* and ϕ.

2 $$E\,I_y\,u'' + P\,u + \phi\left(P\,y_0\right) = 0$$

3 $$E\,I_w\,\phi''' + (P\,\overline{r}_0^{\,2} - G\,K_T)\,\phi' + u'\,(P\,y_0) = 0$$

- These equations will be satisfied by the solutions of the form
- $u = C_2 \sin(\pi z/L)$ and $\phi = C_3 \sin(\pi z/L)$

Column Buckling – Singly Symmetric Columns

$$E\,I_y\,u'' + P\,u + \phi\left(P\,y_0\right) = 0 \qquad \cdots\cdots\cdots(2)$$

$$E\,I_w\,\phi''' + (P\,\bar{r}_0^{\,2} - G\,K_T)\,\phi' + u'\,(P\,y_0) = 0 \cdots\cdots\cdots(3)$$

$$\therefore E\,I_y\,u^{iv} + P\,u'' + \phi''\left(P\,y_0\right) = 0$$

$$E\,I_w\,\phi^{iv} + (P\,\bar{r}_0^{\,2} - G\,K_T)\,\phi'' + u''\,(P\,y_0) = 0$$

$$Let, \quad u = C_2 \sin\frac{\pi z}{L}; \quad \phi = C_3 \sin\frac{\pi z}{L}$$

Therefore, substituting these in equations 2 *and* 3

$$E\,I_y\left(\frac{\pi}{L}\right)^4 C_2 \sin\frac{\pi z}{L} - P\,C_2\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi z}{L} - P\,y_0\left(\frac{\pi}{L}\right)^2 C_3 \sin\frac{\pi z}{L} = 0$$

$$E\,I_w\left(\frac{\pi}{L}\right)^4 C_3 \sin\frac{\pi z}{L} - (P\,\bar{r}_0^{\,2} - G\,K_T)\left(\frac{\pi}{L}\right)^2 C_3 \sin\frac{\pi z}{L} - P\,y_0\left(\frac{\pi}{L}\right)^2 C_2 \sin\frac{\pi z}{L} = 0$$

Column Buckling – Singly Symmetric Columns

$$\therefore \left[E\,I_y \left(\frac{\pi}{L}\right)^2 - P \right] C_2 - P\,y_0\,C_3 = 0$$

$$and \left[E\,I_w \left(\frac{\pi}{L}\right)^2 - (P\,\bar{r}_0^{\,2} - G\,K_T) \right] C_3 - P\,y_0\,C_2 = 0$$

$$Let, P_y = \frac{\pi^2\,E\,I_y}{L^2} \quad and \quad P_\phi = \left(\frac{\pi^2\,E\,I_w}{L^2} + G\,K_T \right) \frac{1}{\bar{r}_0^{\,2}}$$

$$\therefore \left[P_y - P \right] C_2 - P\,y_0\,C_3 = 0$$

$$\left[P_\phi - P \right] \bar{r}_0^{\,2} C_3 - P\,y_0\,C_2 = 0$$

$$\therefore \begin{bmatrix} P_y - P & -P\,y_0 \\ -P\,y_0 & (P_\phi - P)\,\bar{r}_0^{\,2} \end{bmatrix} \begin{Bmatrix} C_2 \\ C_3 \end{Bmatrix} = \{0\}$$

$$\therefore \begin{vmatrix} P_y - P & -P\,y_0 \\ -P\,y_0 & (P_\phi - P)\,\bar{r}_0^{\,2} \end{vmatrix} = 0$$

Column Buckling – Singly Symmetric Columns

$$\therefore (P_y - P)(P_\phi - P)\,\bar{r}_0^{\,2} - P^2\ y_0^2 = 0$$

$$\therefore \left[P_y P_\phi - P(P_y + P_\phi) + P^2 \right] \bar{r}_0^{\,2} - P^2\ y_0^2 = 0$$

$$\therefore P^2 (1 - \frac{y_0^2}{\bar{r}_0^{\,2}}) - P(P_y + P_\phi) + P_y P_\phi = 0$$

$$\therefore P = \frac{(P_y + P_\phi) \pm \sqrt{(P_y + P_\phi)^2 - 4P_y P_\phi (1 - \frac{y_0^2}{\bar{r}_0^{\,2}})}}{2\,(1 - \frac{y_0^2}{\bar{r}_0^{\,2}})}$$

$$\therefore P = \frac{(P_y + P_\phi) \pm \sqrt{(P_y + P_\phi)^2 \left[1 - \frac{4P_y P_\phi (1 - \frac{y_0^2}{\bar{r}_0^{\,2}})}{(P_y + P_\phi)^2} \right]}}{2\,(1 - \frac{y_0^2}{\bar{r}_0^{\,2}})}$$

$$\therefore P = \frac{(P_y + P_\phi)}{2\,(1 - \frac{y_0^2}{\bar{r}_0^{\,2}})} \left[1 \pm \sqrt{1 - \frac{4P_y P_\phi (1 - \frac{y_0^2}{\bar{r}_0^{\,2}})}{(P_y + P_\phi)^2}} \right]$$

Thus, there are two roots for P

Smaller value will govern

$$\therefore P = P = \frac{(P_y + P_\phi)}{2\,(1 - \frac{y_0^2}{\bar{r}_0^{\,2}})} \left[1 - \sqrt{1 - \frac{4P_y P_\phi (1 - \frac{y_0^2}{\bar{r}_0^{\,2}})}{(P_y + P_\phi)^2}} \right]$$

Column Buckling – Singly Symmetric Columns

- The critical buckling load will the lowest of P_x and the two roots shown on the previous slide.
- If the flexural torsional buckling load govern, then the buckling mode will be $C_2 \sin(\pi z/L) \times C_3 \sin(\pi z/L)$
- This buckling mode will include both flexural and torsional deformations – hence flexural-torsional buckling mode.

Column Buckling – Asymmetric Section

- No axes of symmetry: Therefore, shear center S (x_o, y_o) is such that neither x_o not y_o are zero.

$$E\,I_x\,v'' + P\,v - \phi\left(P\,x_0\right) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$E\,I_y\,u'' + P\,u + \phi\left(P\,y_0\right) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$E\,I_w\,\phi''' + (P\,\overline{r}_0^{\,2} - G\,K_T)\,\phi' + u'\,(P\,y_0) - v'\,(P\,x_0) = 0 \qquad \ldots(3)$$

- For simply supported boundary conditions: (*u, u", v, v", ϕ, ϕ"*=0), the solutions to the differential equations can be assumed to be:
  - $u = C_1 \sin(\pi z/L)$
  - $v = C_2 \sin(\pi z/L)$
  - $\phi = C_3 \sin(\pi z/L)$
- These solutions will satisfy the boundary conditions noted above

Column Buckling – Asymmetric Section

- Substitute the solutions into the d.e. and assume that it satisfied too:

$$E\,I_x\left\{-C_1\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi z}{L}\right)\right\}+P\left\{C_1\sin\left(\frac{\pi z}{L}\right)\right\}-P\,x_0\left\{C_3\sin\left(\frac{\pi z}{L}\right)\right\}=0$$

$$E\,I_y\left\{-C_2\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi z}{L}\right)\right\}+P\left\{C_2\sin\left(\frac{\pi z}{L}\right)\right\}+P\,y_0\left\{C_3\sin\left(\frac{\pi z}{L}\right)\right\}=0$$

$$E\,I_w\left\{-C_3\left(\frac{\pi}{L}\right)^3\cos\left(\frac{\pi z}{L}\right)\right\}+(P\,\bar{r}_0^2-G\,K_T)\left\{C_3\frac{\pi}{L}\cos\left(\frac{\pi z}{L}\right)\right\}+P\,y_0\left\{C_1\frac{\pi}{L}\cos\left(\frac{\pi z}{L}\right)\right\}-P\,x_0\left\{C_2\frac{\pi}{L}\cos\left(\frac{\pi z}{L}\right)\right\}=0$$

$$\begin{pmatrix} -\left(\frac{\pi}{L}\right)^2 E\,I_x+P & 0 & -P\,x_0 \\ 0 & -\left(\frac{\pi}{L}\right)^2 E\,I_y+P & P\,y_0 \\ -P\,x_0 & P\,y_0 & -\left(\frac{\pi}{L}\right)^2 E\,I_w+(P\,\bar{r}_0^2-G\,K_T) \end{pmatrix}\begin{bmatrix} C_1\sin\left(\frac{\pi z}{L}\right) \\ C_2\sin\left(\frac{\pi z}{L}\right) \\ \frac{\pi}{L}C_3\cos\left(\frac{\pi z}{L}\right) \end{bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

Column Buckling – Asymmetric Section

$$\begin{pmatrix} -P_x + P & 0 & -P\,x_0 \\ 0 & -P_y + P & P\,y_0 \\ -P\,x_0 & P\,y_0 & \left(-P_\phi + P\right)\bar{r}_0^{\,2} \end{pmatrix} \begin{bmatrix} C_1 \sin\left(\dfrac{\pi z}{L}\right) \\ C_2 \sin\left(\dfrac{\pi z}{L}\right) \\ \dfrac{\pi}{L} C_3 \cos\left(\dfrac{\pi z}{L}\right) \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

where,

$$P_x = \left(\frac{\pi}{L}\right)^2 EI_x \quad P_y = \left(\frac{\pi}{L}\right)^2 EI_y \quad P_\phi = \left(\frac{\pi^2\,E\,I_w}{L^2} + G\,K_T\right)\frac{1}{\bar{r}_0^{\,2}}$$

- Either C_1, C_2, C_3 = 0 (no buckling), or the determinant of the coefficient matrix =0 at buckling.
- Therefore, determinant of the coefficient matrix is:

$$(P - P_x)(P - P_y)(P - P_\phi) - P^2 (P - P_x)\left(\frac{y_o^{\,2}}{\bar{r}_o^{\,2}}\right) - P^2 (P - P_y)\left(\frac{x_o^{\,2}}{\bar{r}_o^{\,2}}\right) = 0$$

Column Buckling – Asymmetric Section

$$(P-P_x)(P-P_y)(P-P_\phi)-P^2(P-P_x)\left(\frac{y_o^{\,2}}{\overline{r}_o^{\,2}}\right)-P^2(P-P_y)\left(\frac{x_o^{\,2}}{\overline{r}_o^{\,2}}\right)=0$$

- This is the equation for predicting buckling of a column with an asymmetric section.
- The equation is cubic in P. Hence, it can be solved to obtain three roots P_{cr1}, P_{cr2}, P_{cr3}.
- The smallest of the three roots will govern the buckling of the column.
- The critical buckling load will always be smaller than P_x, P_y, and P_ϕ
- The buckling mode will always include all three deformations u, v, and ϕ. Hence, it will be a flexural-torsional buckling mode.
- For boundary conditions other than simply-supported, the corresponding P_x, P_y, and P_ϕ can be modified to include end condition effects K_x, K_y, and K_ϕ

Homework No. 4

- See word file
- Problem No. 1
  - Consider a column with doubly symmetric cross-section. The boundary conditions for flexural buckling are simply supported at one end and fixed at the other end.
  - Solve the differential equation for flexural buckling for these boundary conditions and determine the eigenvalue (buckling load) and the eigenmode (buckling shape). Plot the eigenmode.
  - How the eigenvalue compare with the effective length approach for predicting buckling?
  - What is the relationship between the eigenmode and the effective length of the column (Refer textbook).
- Problem No. 2
  - Consider an A992 steel W14 x 68 column cross-section. Develop the normalized buckling load (Pcr/PY) vs. slenderness ratio (L/rx) curves for the column cross-section. Assume that the boundary conditions are simply supported for buckling about the x, y, and z axes.
  - Which buckling mode dominates for different column lengths?
  - Is torsional buckling a possibility for practical columns of this length?
  - Will elastic buckling occur for most practical lengths of this column?
- Problem No. 3
  - Consider a C10 x 30 column section. The length of the column is 15 ft. What is the buckling capacity of the column if it is simply supported for buckling about the y-axis (of non-symmetry), pin-fix for flexure about the x-axis (of symmetry) and simply supported in torsion about the z-axis. Which buckling mode dominates?

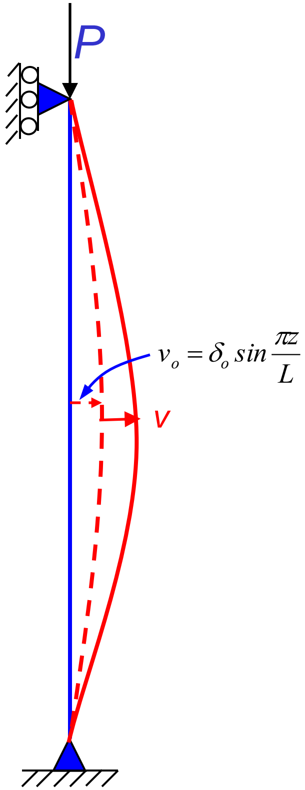
Column Buckling - Inelastic

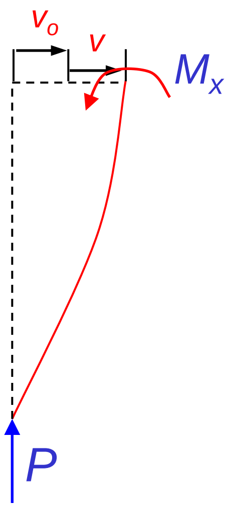
A long topic

Effects of geometric imperfection

$$EI_x v'' + Pv = 0$$
$$EI_y u'' + Pu = 0$$

Leads to bifurcation buckling of perfect doubly-symmetric columns





$$M_x - P(v + v_o) = 0$$
$$\therefore\ EI_x v'' + P(v + v_o) = 0$$
$$\therefore\ v'' + F_v^2 (v + v_o) = 0$$
$$\therefore\ v'' + F_v^2 v = -F_v^2 v_o$$
$$\therefore\ v'' + F_v^2 v = -F_v^2 (\delta_o \sin\frac{\pi z}{L})$$

$$\underline{Solution = v_c + v_p}$$

$$v_c = A\sin(F_v z) + B\cos(F_v z)$$
$$v_p = C\sin\frac{\pi z}{L} + D\cos\frac{\pi z}{L}$$

Effects of Geometric Imperfection

Solve for C and D first

$$\therefore v_p'' + F_v^2 v_p = -F_v^2 \delta_o \sin\frac{\pi z}{L}$$

$$\therefore -\left(\frac{\pi}{L}\right)^2\left[C\sin\frac{\pi z}{L} + D\cos\frac{\pi z}{L}\right] + F_v^2\left[C\sin\frac{\pi z}{L} + D\cos\frac{\pi z}{L}\right] + F_v^2\delta_o\sin\frac{\pi z}{L} = 0$$

$$\therefore \sin\frac{\pi z}{L}\left[-C\left(\frac{\pi}{L}\right)^2 + F_v^2 C + F_v^2\delta_o\right] + \cos\frac{\pi z}{L}\left[-\left(\frac{\pi}{L}\right)^2 D + F_v^2 D\right] = 0$$

$$\therefore -C\left(\frac{\pi}{L}\right)^2 + F_v^2 C + F_v^2\delta_o = 0 \quad and \quad \left[-\left(\frac{\pi}{L}\right)^2 D + F_v^2 D\right] = 0$$

$$\therefore C = \frac{F_v^2\delta_o}{\left(\frac{\pi}{L}\right)^2 - F_v^2} \qquad and \quad D = 0$$

∴ Solution becomes

$$v = A\sin(F_v z) + B\cos(F_v z) + \frac{F_v^2\delta_o}{\left(\frac{\pi}{L}\right)^2 - F_v^2}\sin\frac{\pi z}{L}$$

Geometric Imperfection

$Solve\ for\ A\ and\ B$

$\underline{Boundary\ conditions\ v(0) = v(L) = 0}$

$v(0) = B = 0$

$v(L) = A\sin F_v L = 0$

$\therefore A = 0$

$\underline{\therefore Solution\ becomes}$

$$v = \frac{F_v^2 \delta_o}{\left(\frac{\pi}{L}\right)^2 - F_v^2} \sin\frac{\pi z}{L}$$

$$\therefore v = \frac{\frac{F_v^2}{\left(\frac{\pi}{L}\right)^2}\delta_o}{1 - \frac{F_v^2}{\left(\frac{\pi}{L}\right)^2}} \sin\frac{\pi z}{L} = \frac{\frac{P}{P_E}\delta_o}{1 - \frac{P}{P_E}} \sin\frac{\pi z}{L}$$

$$\therefore v = \frac{\frac{P}{P_E}}{1 - \frac{P}{P_E}} \delta_o \sin\frac{\pi z}{L}$$

$\underline{\therefore Total\ Deflection}$

$$= v + v_o = \frac{\frac{P}{P_E}}{1 - \frac{P}{P_E}} \delta_o \sin\frac{\pi z}{L} + \delta_o \sin\frac{\pi z}{L}$$

$$= \left[\frac{\frac{P}{P_E}}{1 - \frac{P}{P_E}} + 1\right] \delta_o \sin\frac{\pi z}{L} = \frac{1}{1 - \frac{P}{P_E}} \delta_o \sin\frac{\pi z}{L}$$

$$= A_F \delta_o \sin\frac{\pi z}{L}$$

A_F = amplification factor

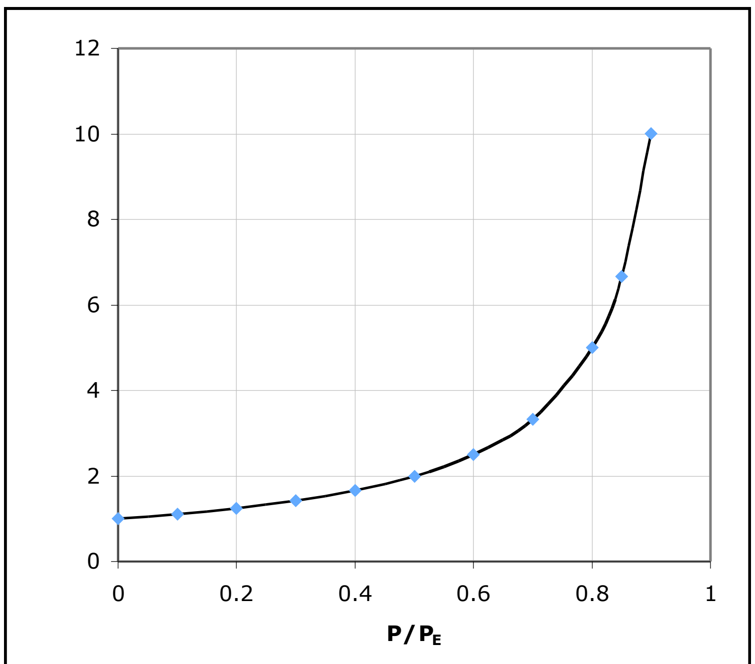
Geometric Imperfection

$$A_F = \frac{1}{1 - \dfrac{P}{P_E}} = amplification \;\; factor$$

$$M_x = P(v + v_o)$$

$$\therefore M_x = A_F (P \delta_o \, sin \frac{\pi z}{L})$$

$$i.e., M_x = A_F \times (moment \; due \; to \; initial \; crooked)$$



Increases exponentially
Limit A_F for design
Limit P/P_E for design

Value used in the code is 0.877
This will give A_F = 8.13
Have to live with it.

Residual Stress Effects

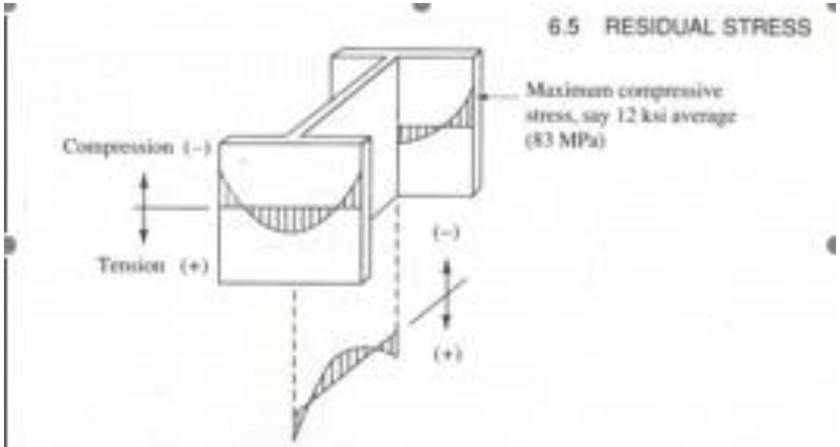


Figure 6.5.1 Typical residual stress pattern on rolled shapes.

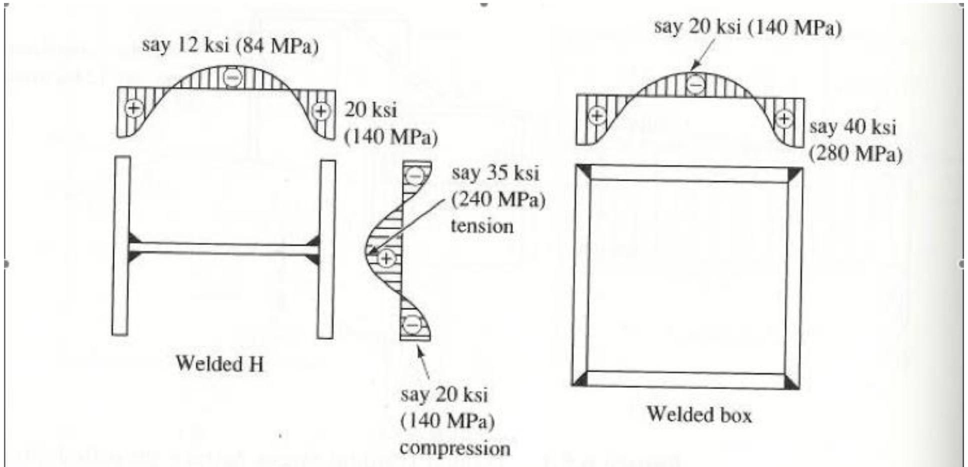


Figure 6.5.3 Typical residual stress distribution in welded shapes.

Residual Stress Effects

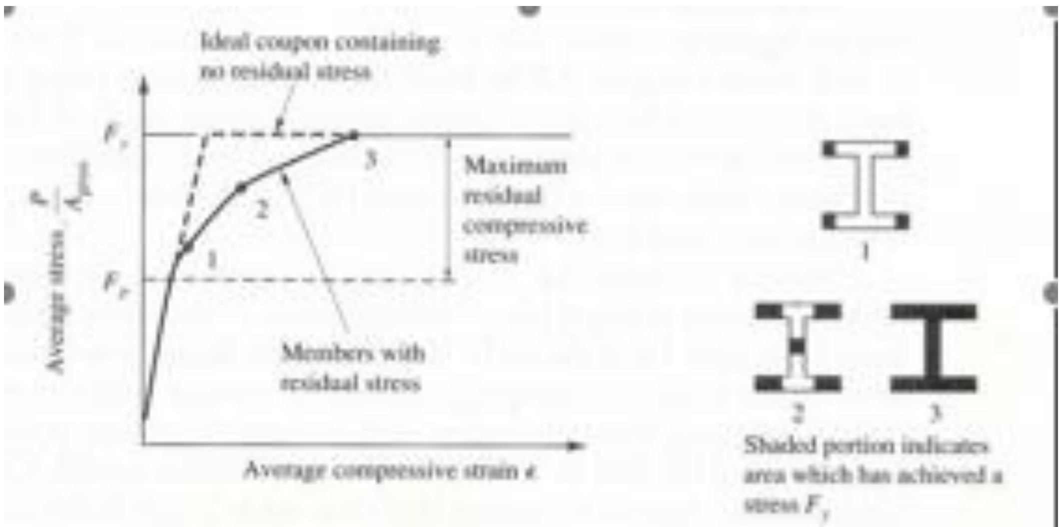
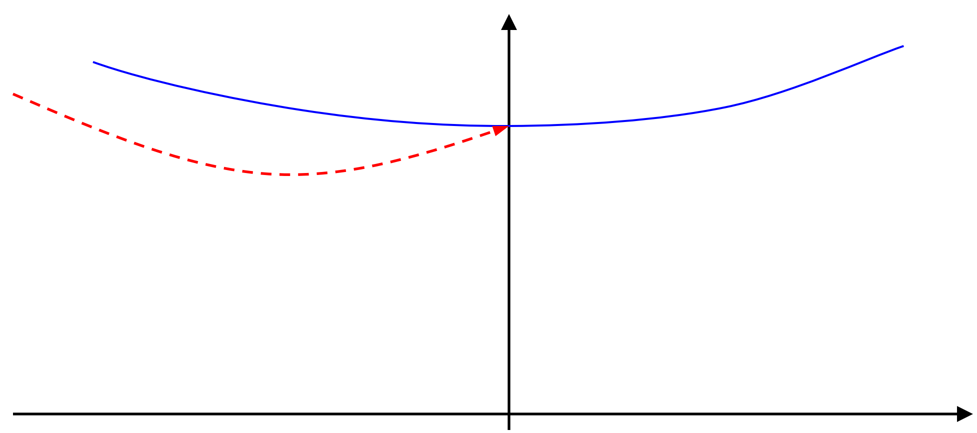


Figure 6.5.2 Influence of residual stress on average stress-strain curve.

History of column inelastic buckling

- Euler developed column elastic buckling equations (buried in the million other things he did).
  - Take a look at: http://en.wikipedia.org/wiki/EuleR
  - An amazing mathematician
- In the 1750s, I could not find the exact year.
- The elastica problem of column buckling indicates elastic buckling occurs with no increase in load.
  - dP/dv=0



History of Column Inelastic Buckling

- Engesser extended the elastic column buckling theory in 1889.
- He assumed that inelastic buckling occurs with no increase in load, and the relation between stress and strain is defined by tangent modulus E_t

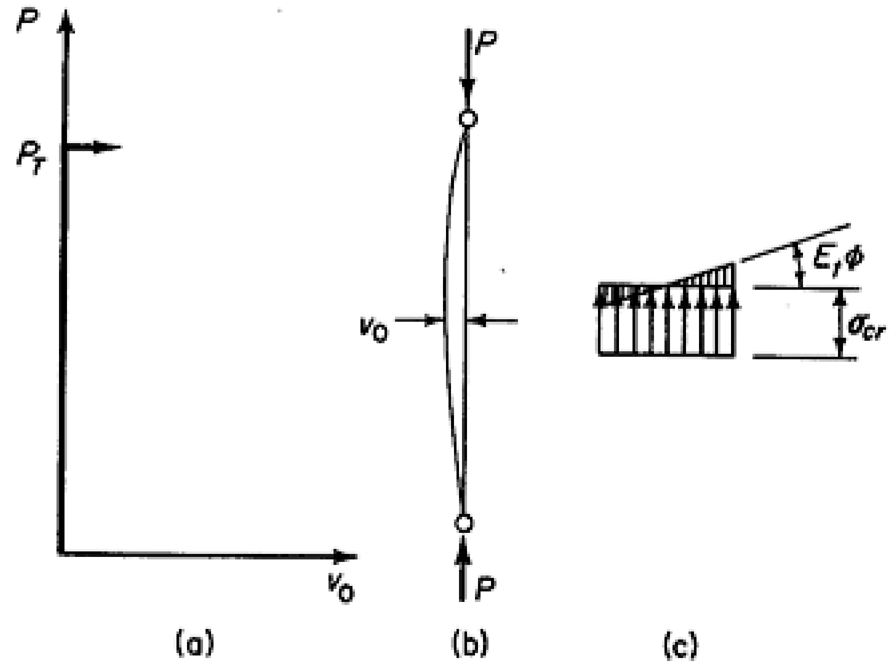
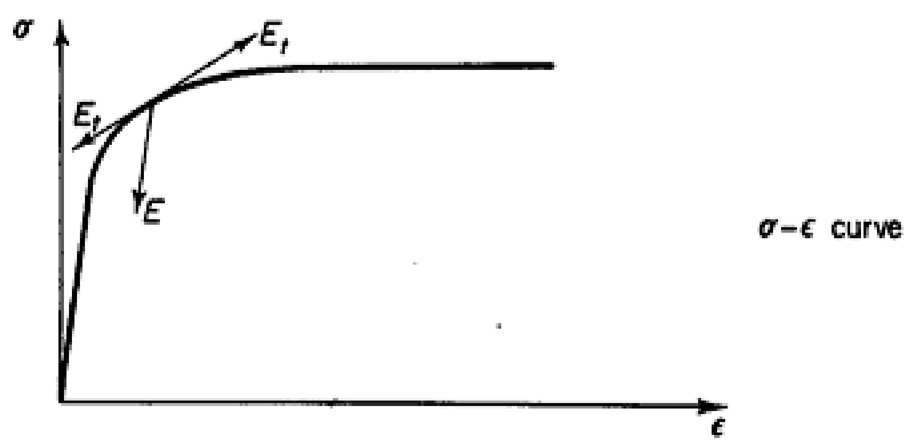


Fig. 4.21. Engesser's concept of inelastic column buckling

- Engesser's tangent modulus theory is easy to apply. It compares reasonably with experimental results.
  - $P_T = \pi E_T I / (KL)^2$

History of Column Inelastic Buckling

- In 1895, Jasinsky pointed out the problem with Engesser's theory.
  - If dP/dv=0, then the 2nd order moment (*Pv*) will produce incremental strains that will vary linearly and have a zero value at the centroid (neutral axis).
  - The linear strain variation will have compressive and *tensile* values. The tangent modulus for the incremental compressive strain is equal to E_t and that for the tensile strain is E.



History of Column Inelastic Buckling

- In 1898, Engesser corrected his original theory by accounting for the different tangent modulus of the tensile increment.
  - This is known as the *reduced modulus* or *double modulus*
  - The assumptions are the same as before. That is, there is no increase in load as buckling occurs.
- The corrected theory is shown in the following slide

History of Column Inelastic Buckling

- The buckling load P_R produces critical stress $\sigma_R = P_r/A$
- During buckling, a small curvature $d\phi$ is introduced
- The strain distribution is shown.
- The loaded side has $d\varepsilon_L$ and $d\sigma_L$
- The unloaded side has $d\varepsilon_U$ and $d\sigma_U$

$$d\varepsilon_L = (\bar{y} - y_1 + y)\, d\phi$$

$$d\varepsilon_U = (y - \bar{y} + y_1)\, d\phi$$

$$\therefore d\sigma_L = E_t(\bar{y} - y_1 + y)\, d\phi$$

$$\therefore d\sigma_U = E(y - \bar{y} + y_1)\, d\phi$$

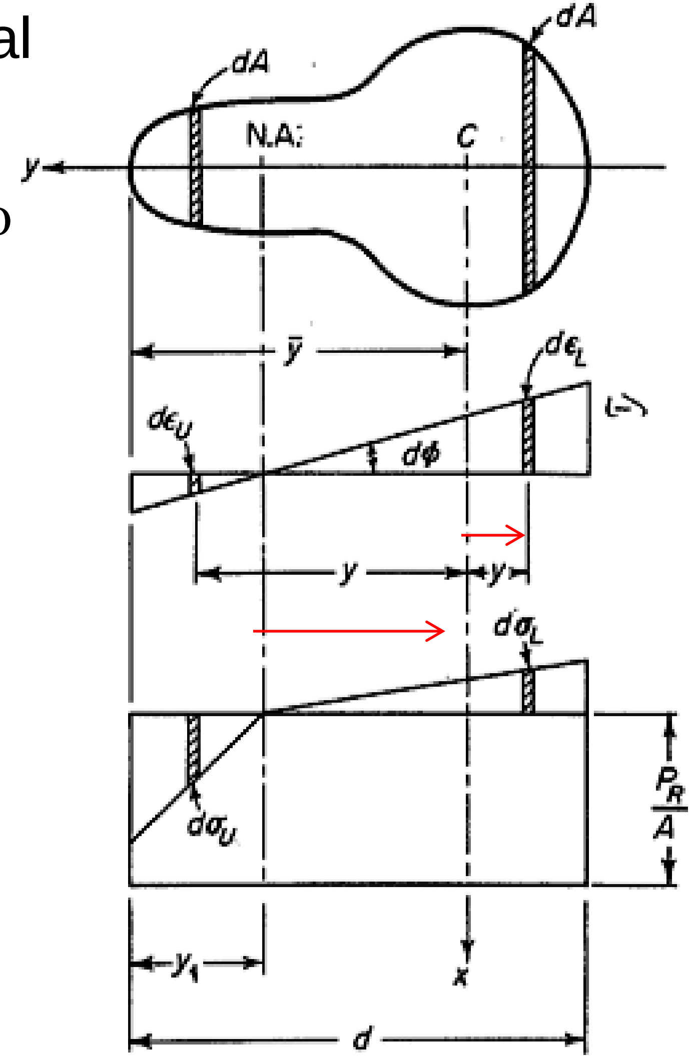


Fig. 4.22. The reduced modulus conce

History of Column Inelastic Buckling

$$\because\ d\phi = -v''$$

$$d\sigma_L = -E_t(\bar{y} - y_1 + y)\, v''$$

$$d\sigma_U = -E(y - \bar{y} + y_1)\, v''$$

$$But,\ the\ assumption\ is\ dP = 0$$

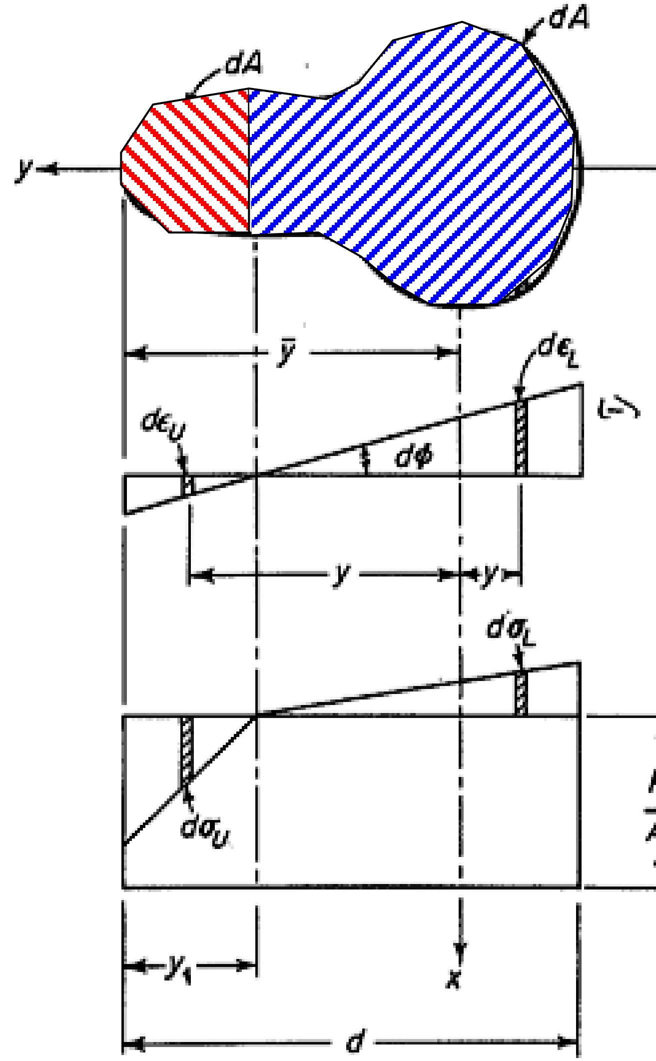
$$\therefore \int_{\bar{y}-y_1}^{\bar{y}} d\sigma_U\, dA - \int_{-(d-\bar{y})}^{\bar{y}-y_1} d\sigma_L\, dA = 0$$

$$\therefore \int_{\bar{y}-y_1}^{\bar{y}} E(y - \bar{y} + y_1)\, dA - \int_{-(d-\bar{y})}^{\bar{y}-y_1} E_t(\bar{y} - y_1 + y)\, dA = 0$$

$$\therefore ES_1 - E_t S_2 = 0$$

$$where,\ S_1 = \int_{\bar{y}-y_1}^{\bar{y}} (y - \bar{y} + y_1)\, dA$$

$$and\quad S_2 = \int_{-(d-\bar{y})}^{\bar{y}-y_1} (\bar{y} - y_1 + y)\, dA$$



History of Column Inelastic Buckling

- S_1 and S_2 are the statical moments of the areas to the left and right of the neutral axis.
  - Note that the neutral axis does not coincide with the centroid any more.
  - The location of the neutral axis is calculated using the equation derived $ES_1 - E_tS_2 = 0$

$$M = Pv$$

$$\therefore M = \int_{\bar{y}-y_1}^{\bar{y}} d\sigma_U (y - \bar{y} + y_1)\, dA - \int_{-(d-\bar{y})}^{\bar{y}-y_1} d\sigma_L (\bar{y} - y_1 + y)\, dA$$

$$\therefore M = Pv = -v''(EI_1 + E_t I_2)$$

$$where,\ I_1 = \int_{\bar{y}-y_1}^{\bar{y}} (y - \bar{y} + y_1)^2\, dA$$

$$and \quad I_2 = \int_{-(d-\bar{y})}^{\bar{y}-y_1} (\bar{y} - y_1 + y)^2\, dA$$

History of Column Inelastic Buckling

$$M = Pv = -v''(EI_1 + E_t I_2)$$

$$\therefore Pv + (EI_1 + E_t I_2)v'' = 0$$

$$\therefore v'' + \frac{P}{EI_1 + E_t I_2} v = 0$$

$$\therefore v'' + F_v^2 v = 0$$

$$where, \quad F_v^2 = \frac{P}{EI_1 + E_t I_2} = \frac{P}{\overline{E} I_x}$$

$$and \ \overline{E} = E\frac{I_1}{I_x} + E_t \frac{I_2}{I_x}$$

$$P_R = \frac{\pi^2 \overline{E} I_x}{(KL)^2}$$

$\underline{E}$ is the reduced or double modulus

P_R is the reduced modulus buckling load

History of Column Inelastic Buckling

- For 50 years, engineers were faced with the dilemma that the reduced modulus theory is correct, but the experimental data was closer to the tangent modulus theory. How to resolve?
- Shanley eventually resolved this dilemma in 1947. He conducted very careful experiments on small aluminum columns.
  - He found that lateral deflection started very near the theoretical tangent modulus load and the load capacity increased with increasing lateral deflections.
  - The column axial load capacity never reached the calculated reduced or double modulus load.
- Shanley developed a column model to explain the observed phenomenon

History of Column Inelastic Buckling

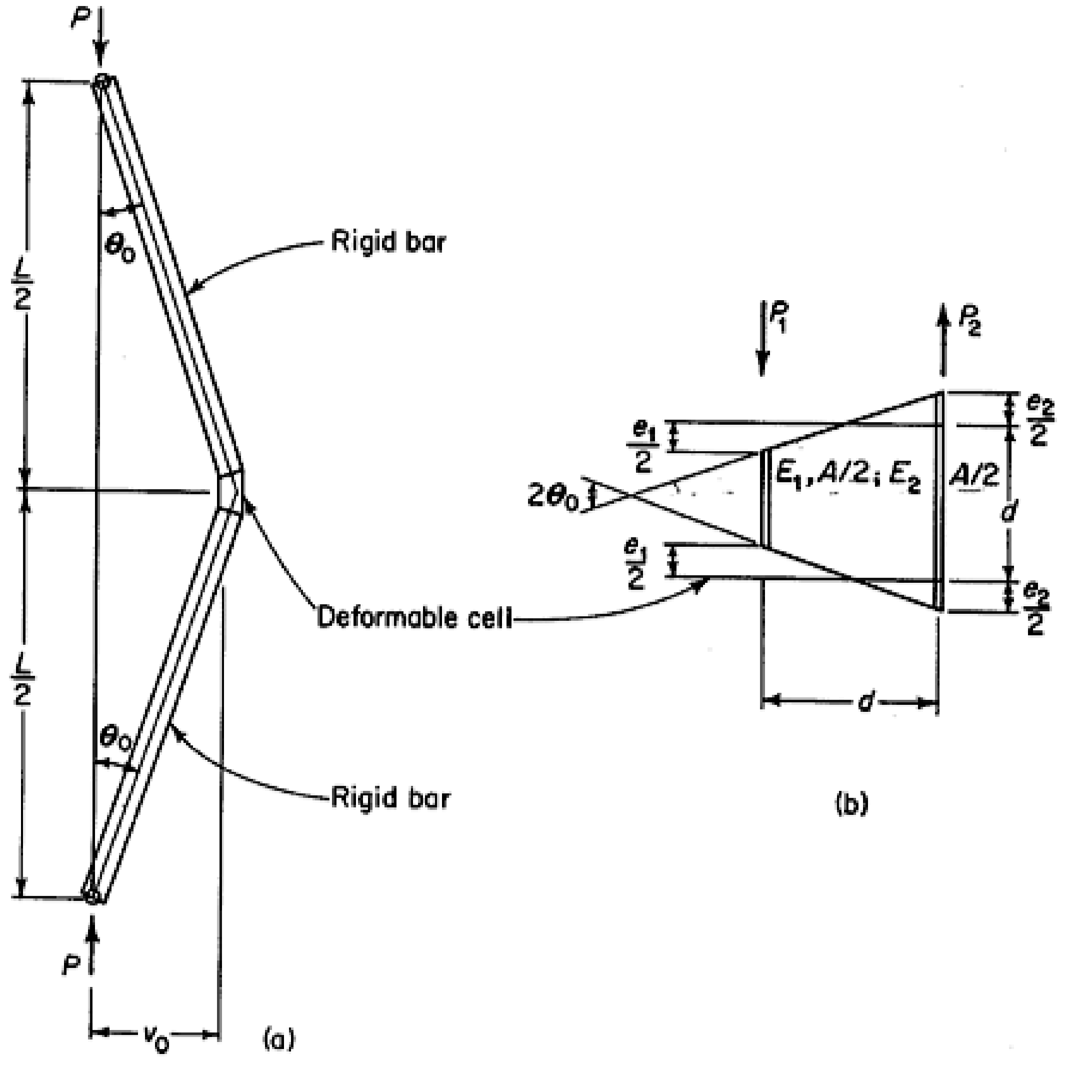


Fig. 4.23. Shanley's column model

History of Column Inelastic Buckling

$$v_0 = \frac{\theta_0 L}{2} \quad \text{and} \quad \theta_0 = \frac{1}{2d}(e_1 + e_2) \tag{4.129}$$

By combining these two equations we can eliminate θ_0, and thus

$$v_0 = \frac{L}{4d}(e_1 + e_2) \tag{4.130}$$

The external moment at the midheight of the column is

$$M_e = Pv_0 = \frac{PL}{4d}(e_1 + e_2) \tag{4.131}$$

The forces in the two flanges due to buckling are

$$P_1 = \frac{E_1 e_1 A}{2d} \quad \text{and} \quad P_2 = \frac{E_2 e_2 A}{2d} \tag{4.132}$$

The internal moment is then

$$M_i = \frac{dP_1}{2} + \frac{dP_2}{2} = \frac{A}{4}(E_1 e_1 + E_2 e_2) \tag{4.133}$$

With $M_e = M_i$ we get an expression for the axial load P, or

$$P = \frac{Ad}{L}\left(\frac{E_1 e_1 + E_2 e_2}{e_1 + e_2}\right) \tag{4.134}$$

History of Column Inelastic Buckling

In case the cell is elastic $E_1 = E_2 = E$, and so

$$P_E = \frac{AEd}{L} \tag{4.135}$$

For the tangent modulus concept $E_1 = E_2 = E_t$, and so

$$P_T = \frac{AE_t d}{L} \tag{4.136}$$

When we consider the elastic unloading of the "tension" flange, then $E_1 = E_t$ and $E_2 = E$, and thus

$$P = \frac{Ad}{L}\left(\frac{E_t e_1 + E_2 e_2}{e_1 + e_2}\right) \tag{4.137}$$

Upon substitution of e_1 from Eq. (4.130) and P_T from Eq. (4.136) and using the abbreviation

$$\tau = \frac{E_t}{E} \tag{4.138}$$

we find that

$$P = P_T\left[1 + \frac{Le_2}{4dv_0}\left(\frac{1}{\tau} - 1\right)\right] \tag{4.139}$$

History of Column Inelastic Buckling

$$P = P_T\left[1 + \frac{1}{(d/2v_0) + (1+\tau)/(1-\tau)}\right] \quad (4.143)$$

$$P_R = P_T\left(1 + \frac{1-\tau}{1+\tau}\right) \quad (4.146)$$

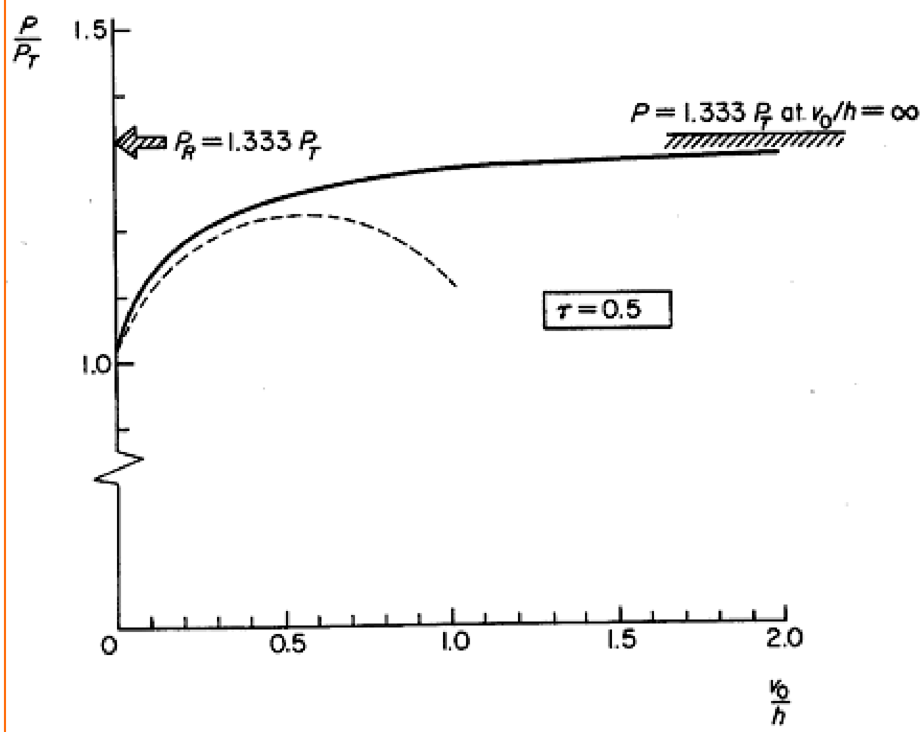
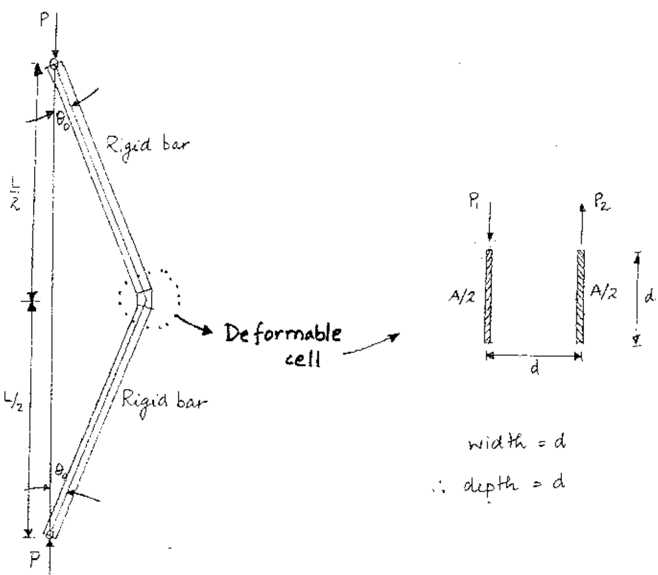


Fig. 4.24. Post-buckling behavior in the inelastic range

2.3.3 INELASTIC COLUMNS: Stage III - Shanley's Contribution

- Shanley (1947) conducted very careful tests on small aluminum columns. He found that:
  - lateral deflections (v) started very near the tangent modulus load P_T
  - but, additional load was carried until unloading set in.
  - The reduced modulus P_R could never be reached.

- Shanley's explanation:



width = d

∴ depth = d

$$\therefore M_{ext} = P \times v_0 = P \times \theta_0 \times \frac{L}{2} \quad \ldots\ldots\ldots \quad (23)$$

The moment M_{ext} produces strains & stresses in the deformable cell

$\therefore \phi$ = curvature of cell $= \dfrac{2\theta_0}{d}$

and $\phi = \dfrac{\varepsilon_1 + \varepsilon_2}{d}$

$$\therefore \theta_0 = \frac{\varepsilon_1 + \varepsilon_2}{2} \quad \ldots$$

$$\& \quad v_0 = \frac{\varepsilon_1 + \varepsilon_2}{2} \times \frac{L}{2}$$

where $\varepsilon_1 \equiv$ strain in compressed fiber

$\varepsilon_2 \equiv$ strain in tension flange

Now, P_1 = force in compression flange $= \frac{A}{2} \times E_1 \times \varepsilon_1$

P_2 = force in tension flange $= \frac{A}{2} \times E_2 \times \varepsilon_2$

$$\therefore P_1 - P_2 = \frac{A}{2} \times \{E_1 \varepsilon_1 - E_2 \varepsilon_2\} \quad \ldots\ldots\ldots \quad (25)$$

$$M_{int} = \frac{P_1 + P_2}{2} \times d = \frac{Ad}{4} \times \{E_1 \varepsilon_1 + E_2 \varepsilon_2\} \quad \ldots\ldots \quad (26)$$

But $M_{ext} = M_{int}$

$$\therefore P \times \theta_0 \times \frac{L}{2} = \frac{Ad}{4} \times \{E_1 \varepsilon_1 + E_2 \varepsilon_2\}$$

$$\therefore P \times \frac{(\varepsilon_1 + \varepsilon_2)}{2} \times \frac{L}{2} = \frac{Ad}{4} \times \{E_1 \varepsilon_1 + E_2 \varepsilon_2\}$$

$$\therefore P = \frac{Ad}{L} \times \left\{\frac{E_1 \varepsilon_1 + E_2 \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right\} \quad \ldots\ldots\ldots \quad (27)$$

∴ if the cell is elastic: $E_1 = E_2 = E$

$$P_E = \frac{Ad}{L} \times E$$

∴ if the cell is inelastic with $E_1 = E_2 = E_t$

then $P_T = \frac{Ad}{L} \times E_t$ ⟶ (28)

& if $E_1 = E_t$ and $E_2 = E$

then
$$P = \frac{Ad}{L} \times \left\{ \frac{E_t \varepsilon_1 + E \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \right\}$$
$$= \frac{Ad}{L} \times \left\{ E_t + (E - E_t) \times \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \right\}$$
$$\therefore P = \frac{Ad}{L} \times E_t \left\{ 1 + \left(\frac{1}{\tau} - 1\right) \times \frac{\varepsilon_2}{\frac{4 v_0}{L}} \right\} \quad \cdots\cdots \rightarrow \quad \tau = \frac{E_t}{E}, \quad \varepsilon_1 + \varepsilon_2 = \frac{4 v_0}{L}$$
$$\therefore P = P_T \left\{ 1 + \left(\frac{1}{\tau} - 1\right) \times \frac{L \varepsilon_2}{4 v_0} \right\} \quad \cdots\cdots\cdots \quad (29)$$

Additionally:

$$P = P_T + P_1 - P_2$$
$$= \frac{Ad\,E_t}{L} + \frac{A}{2} \times \left\{ E_t \varepsilon_1 - E \varepsilon_2 \right\}$$
$$= \frac{Ad}{L} E_t + \frac{A}{2} \times E_t (\varepsilon_1 + \varepsilon_2) - \frac{A}{2} (E + E_t) \varepsilon_2$$
$$P = \frac{Ad}{L} E_t \times \left\{ 1 + \frac{2 v_0}{d} - \frac{L \varepsilon_2}{2d} \left(\frac{1}{\tau} + 1 \right) \right\}$$
$$P = P_T \left\{ 1 + \frac{2 v_0}{d} - \frac{L \varepsilon_2}{2d} \left(\frac{1}{\tau} + 1 \right) \right\} \quad \cdots\cdots\cdots \quad (30)$$

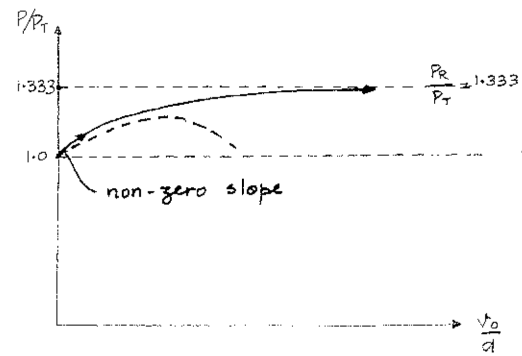
Using equations (29) & (30) to eliminate ε_2

$$P = P_T \times \left\{ 1 + \frac{1}{\frac{d}{2 v_0} + (1+\tau)(1-\tau)} \right\} \longrightarrow (31)$$

For example;

if $\tau = 0.5$ then
$$P = P_T \times \left\{ 1 + \frac{1}{\frac{d}{2 v_0} + 3} \right\} \longrightarrow (32)$$

The plot of $\frac{P}{P_T}$ vs. $\frac{v_0}{d}$ ⟶ shown below



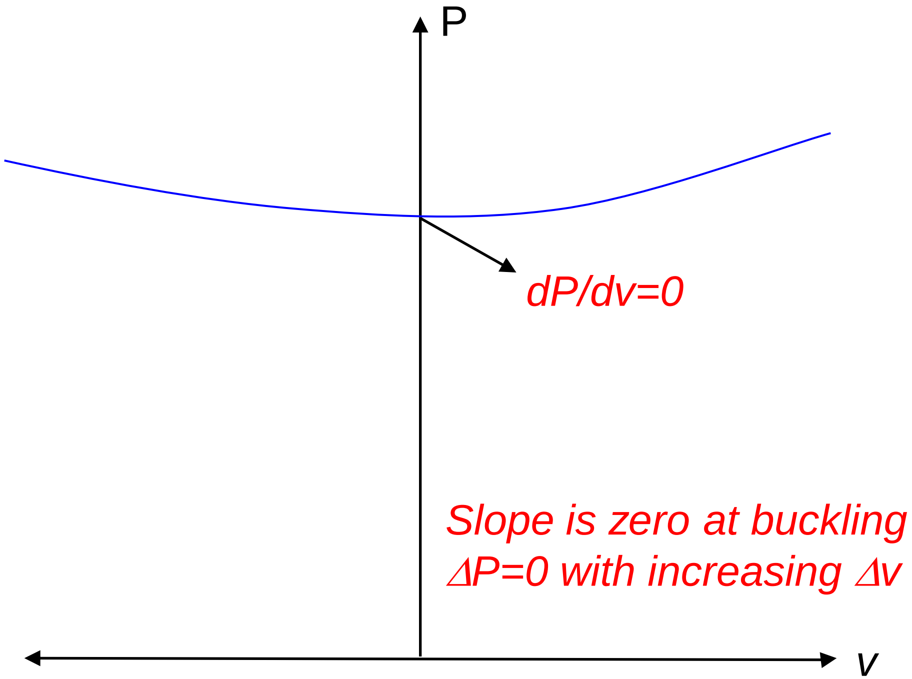
- lateral deflections occur when P_T is reached
- buckling occurs with increasing loads
- curve approaches P_R as $\frac{v_0}{d} \rightarrow \infty$
- If τ decreases with strain ⟶ P_R will never be reached and the dotted curve will be followed

Then $P_T \leq P_{max} < P_R$

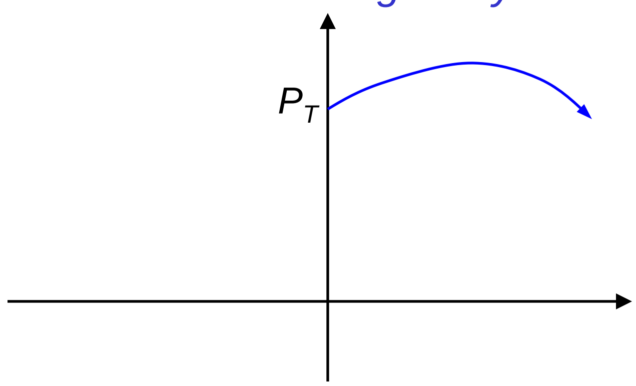
Column Inelastic Buckling

- Three different theories
  - Tangent modulus
  - Reduced modulus
  - Shanley model

- Tangent modulus theory assumes
  - Perfectly straight column
  - Ends are pinned
  - Small deformations
  - No strain reversal during buckling

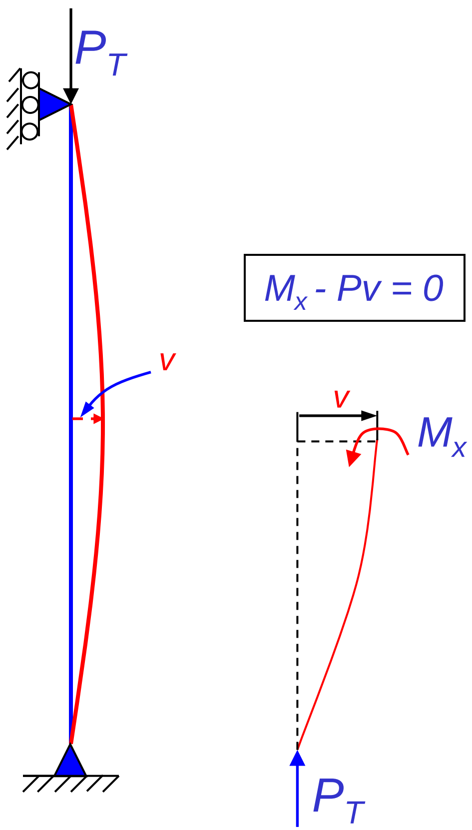


Elastic buckling analysis

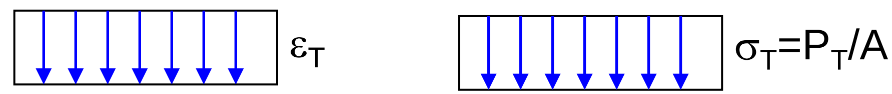


Tangent modulus theory

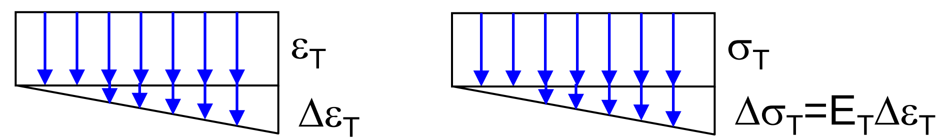
- Assumes that the column buckles at the tangent modulus load such that there is an increase in ΔP (axial force) and ΔM (moment).
  - The axial strain increases everywhere and there is no strain reversal.



Strain and stress state just before buckling



Strain and stress state just after buckling



Curvature = ϕ = slope of strain diagram

$$\therefore \phi = \frac{\Delta\varepsilon_T}{h}$$

$$\Delta\varepsilon_T = \phi\left(\frac{h}{2} + y\right) \qquad \text{where } y = \text{distance from centroid}$$

$$\Delta\sigma_T = \phi\left(\frac{h}{2} + y\right) \bullet E_T$$

Tangent modulus theory

- Deriving the equation of equilibrium

- The equation $M_x - P_T v=0$ becomes $-E_T I_x v'' - P_T v=0$
  - Solution is $P_T = \pi^2 E_T I_x / L^2$

Example - Aluminum columns

- Consider an aluminum column with Ramberg-Osgood stress-strain curve

$$\varepsilon = \frac{\sigma}{E} + 0.002\left(\frac{\sigma}{\sigma_{0.2}}\right)^n$$

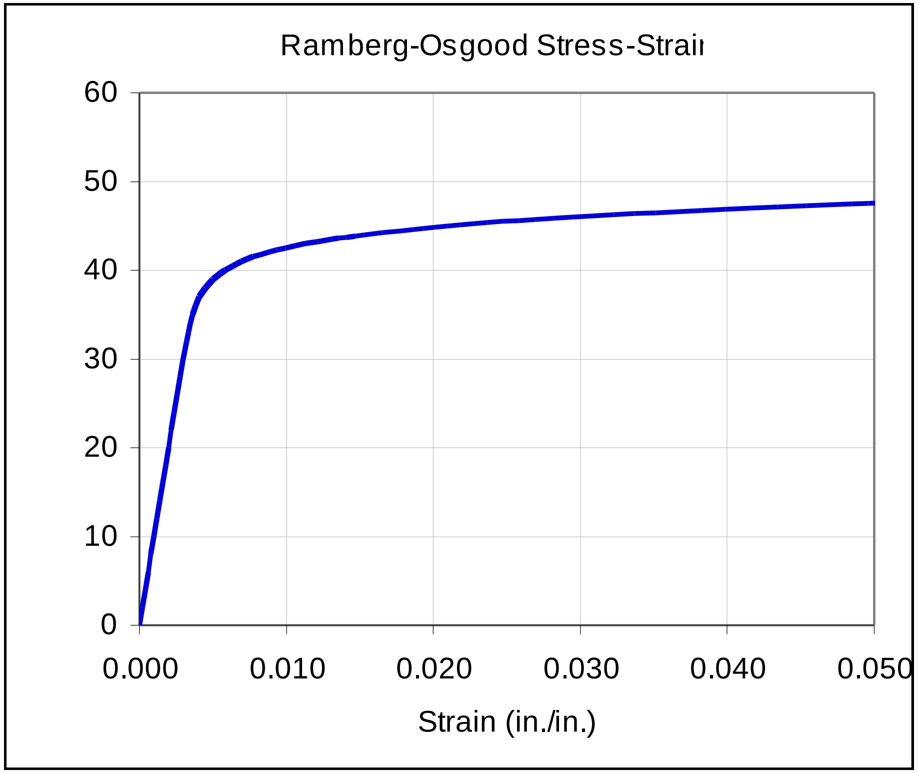
$$\therefore \frac{\partial \varepsilon}{\partial \sigma} = \frac{1}{E} + \frac{0.002}{\sigma_{0.2}^n} n \sigma^{n-1}$$

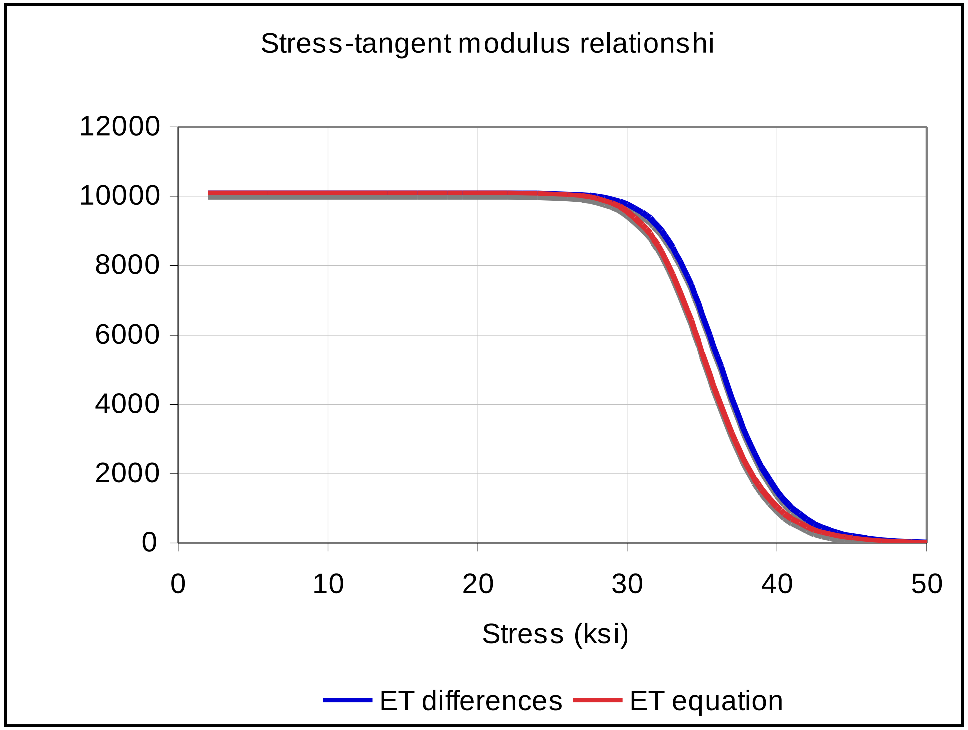
$$\therefore \frac{\partial \varepsilon}{\partial \sigma} = \frac{1 + \frac{0.002}{\sigma_{0.2}^n} n E \sigma^{n-1}}{E}$$

$$\therefore \frac{\partial \varepsilon}{\partial \sigma} = \frac{1 + \frac{0.002}{\sigma_{0.2}} n E \left(\frac{\sigma}{\sigma_{0.2}}\right)^{n-1}}{E}$$

$$\therefore \frac{\partial \sigma}{\partial \varepsilon} = \frac{E}{1 + \frac{0.002}{\sigma_{0.2}} n E \left(\frac{\sigma}{\sigma_{0.2}}\right)^{n-1}} = E_T$$

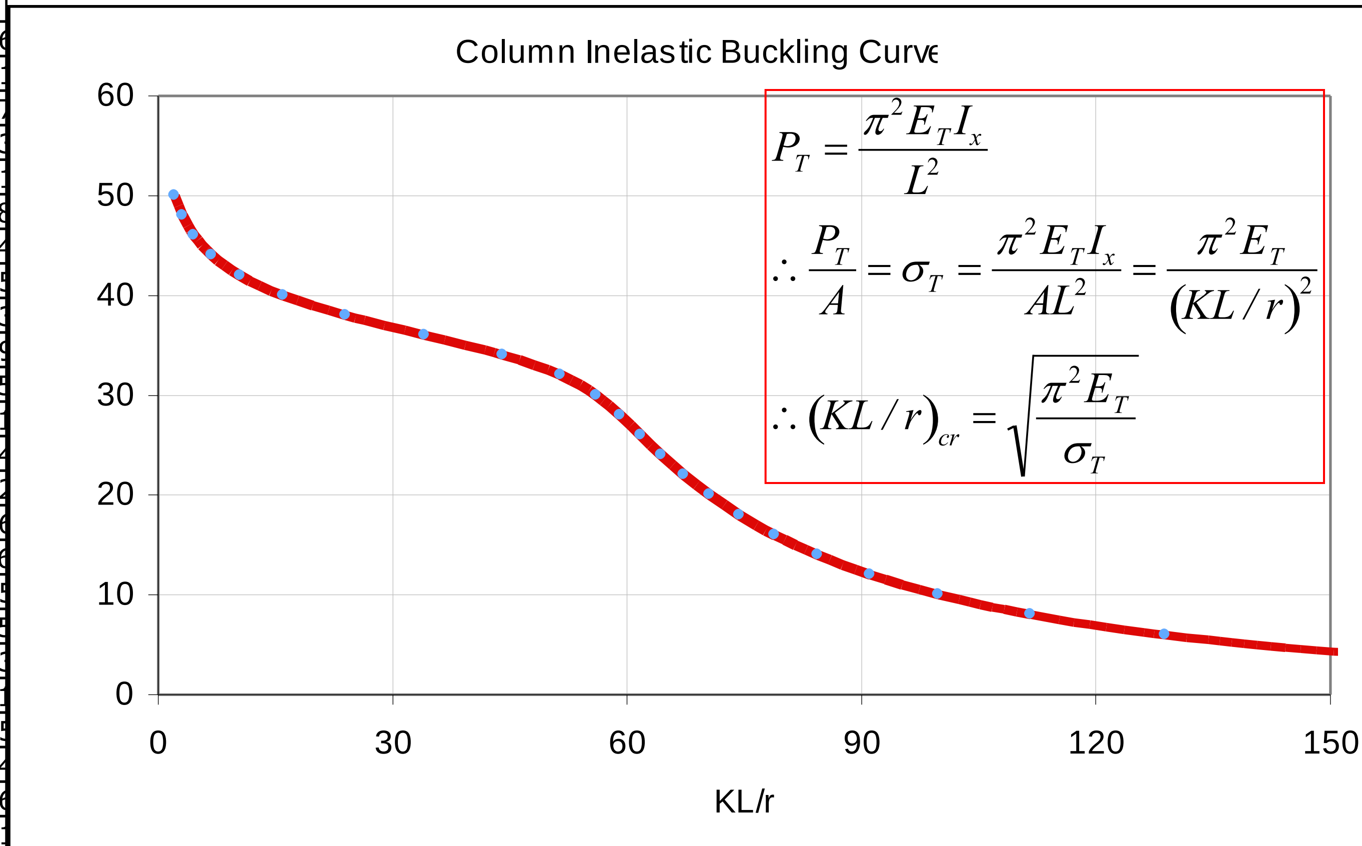
Tangent Modulus Buckling





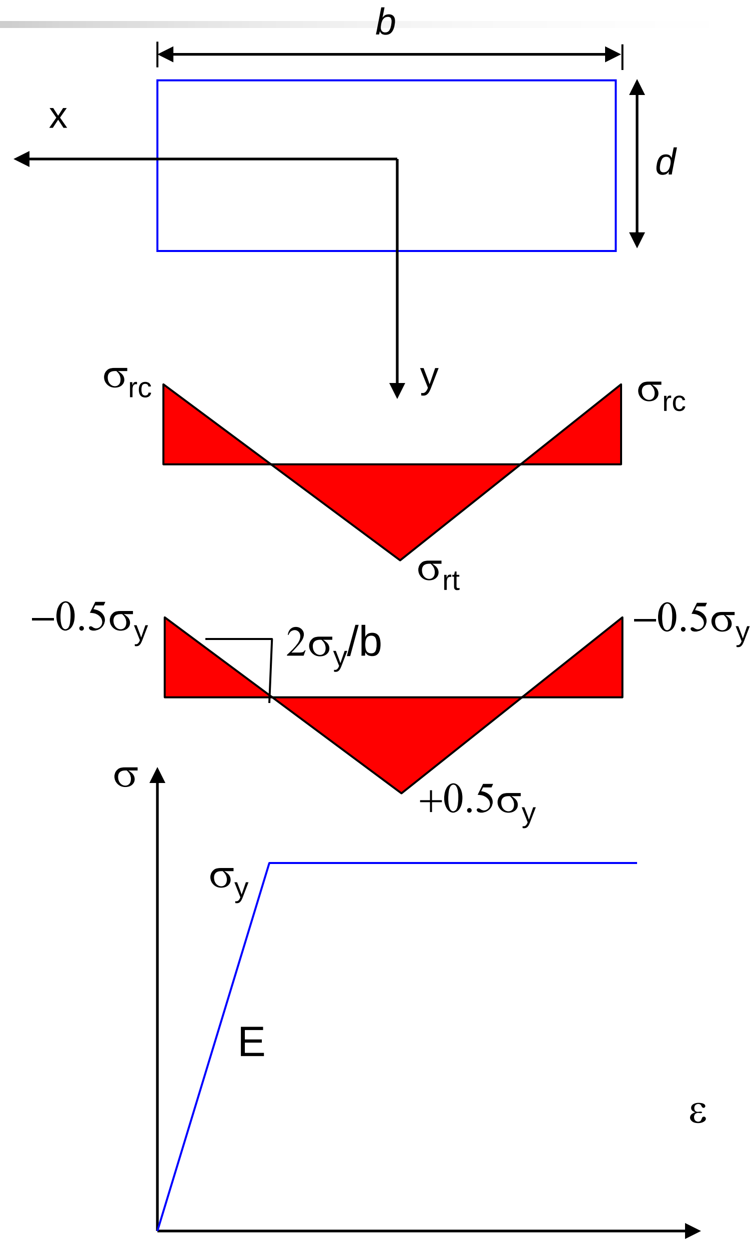
Tangent Modulus Buckling

σ_T	$(KL/r)_{cr}$
0	
2	223.2521046
4	157.8630771
6	128.8946627
8	111.6260523
10	99.84137641
12	91.1422898
14	84.3813604
16	78.9315027
18	74.4171015
20	70.5969067
22	67.3048795
24	64.4113691
26	61.7785743
28	59.1743095
30	56.0920828
32	51.5097656
34	44.1456641
36	34.1419685
38	24.0046401
40	15.9961201
42	10.4882747
44	6.90251614
46	4.59663340
48	3.10544036
50	2.12914520



Residual Stress Effects

- Consider a rectangular section with a simple residual stress distribution
- Assume that the steel material has elastic-plastic stress-strain $\sigma-\varepsilon$ curve.
- Assume simply supported end conditions
- Assume triangular distribution for residual stresses



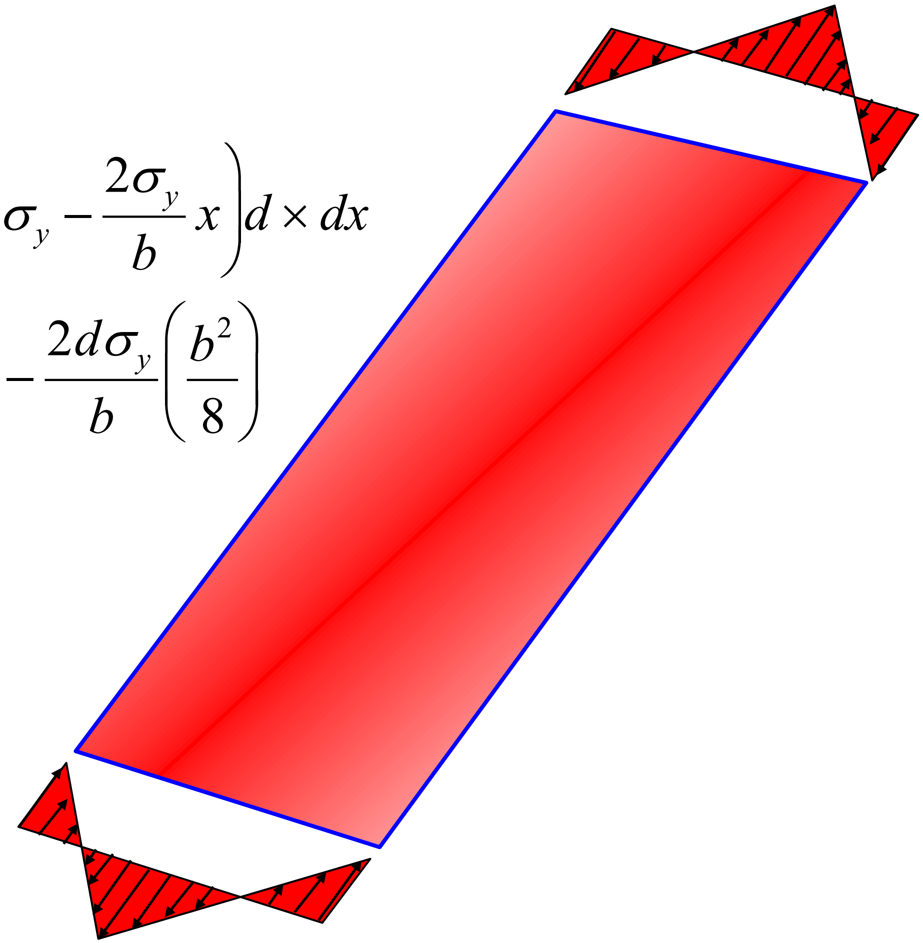
Residual Stress Effects

- One major constrain on residual stresses is that they must be such that

$$\therefore \int_{-b/2}^{0}\left(-0.5\sigma_y+\frac{2\sigma_y}{b}x\right)d\times dx+\int_{0}^{b/2}\left(+0.5\sigma_y-\frac{2\sigma_y}{b}x\right)d\times dx$$

$$=-0.5\sigma_y d\,b/2+0.5\sigma_y d\,b/2+\frac{2d\sigma_y}{b}\left(\frac{b^2}{8}\right)-\frac{2d\sigma_y}{b}\left(\frac{b^2}{8}\right)$$

$$=0$$



- Residual stresses are produced by uneven cooling but no load is present

Residual Stress Effects

- Response will be such that - elastic behavior when

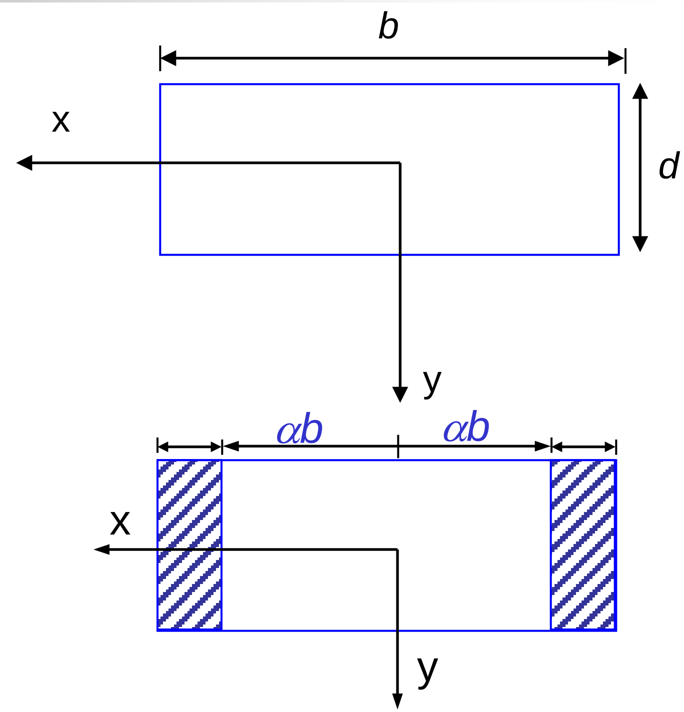
$$\sigma < 0.5\sigma_y$$

$$P_x = \frac{\pi^2 EI_x}{L^2} \quad and \quad P_y = \frac{\pi^2 EI_y}{L^2}$$

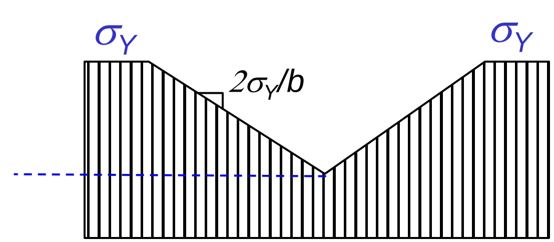
Yielding occurs when

$$\sigma = 0.5\sigma_y \quad i.e., P = 0.5P_Y$$

Inelastic buckling will occur after $\sigma > 0.5\sigma_y$



$$\left(\sigma_Y - \frac{2\sigma_Y}{b}\alpha b\right) = \sigma_Y(1-2\alpha)$$



Residual Stress Effects

$$Total\ axial\ force\ corresponding\ to\ the\ yielded\ section$$

$$\sigma_Y(b-2\alpha b)d+\left(\frac{\sigma_Y+\sigma_Y(1-2\alpha)}{2}\right)\alpha bd\times 2$$

$$=\sigma_Y(1-2\alpha)bd+\sigma_Y(2-2\alpha)\alpha bd$$

$$=\sigma_Y bd-2\alpha bd\sigma_Y+2\sigma_Y\alpha bd-2\alpha^2 bd\sigma_Y$$

$$=\sigma_Y bd(1-2\alpha^2)=P_Y(1-2\alpha^2)$$

$$\therefore If\ inelastic\ buckling\ were\ to\ occur\ at\ this\ load$$

$$P_{cr}=P_Y(1-2\alpha^2)$$

$$\therefore \alpha=\sqrt{\frac{1}{2}\left(1-\frac{P_{cr}}{P_Y}\right)}$$

$If\ inelastic\ buckling\ occurs\ about\ x-axis$

$$P_{cr} = P_{Tx} = \frac{\pi^2 E}{L^2}(2\alpha b)\frac{d^3}{12}$$

$$\therefore P_{Tx} = \frac{\pi^2 E I_x}{L^2} 2\alpha$$

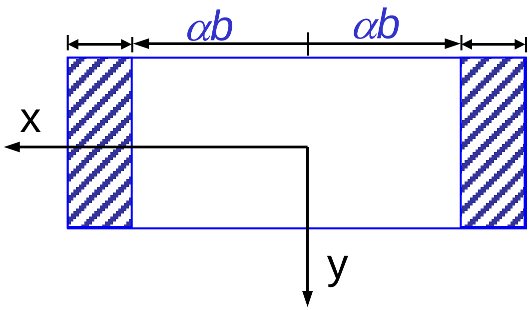
$$\therefore P_{Tx} = P_x \times 2 \times \sqrt{\frac{1}{2}\left(1 - \frac{P_{cr}}{P_Y}\right)}$$

$$\therefore P_{Tx} = P_x \times 2 \times \sqrt{\frac{1}{2}\left(1 - \frac{P_{Tx}}{P_Y}\right)} \qquad \because P_{cr} = P_{Tx}$$

$$\therefore \frac{P_{Tx}}{P_Y} = \frac{P_x}{P_Y} \times 2 \times \sqrt{\frac{1}{2}\left(1 - \frac{P_{Tx}}{P_Y}\right)} \qquad Let, \frac{P_x}{P_Y} = \frac{1}{\lambda_x^2} = \pi^2 \frac{E}{\sigma_Y}\left(\frac{r_x}{K_x L_x}\right)^2$$

$$\therefore \frac{P_{Tx}}{P_Y} = \frac{1}{\lambda_x^2} \times 2 \times \sqrt{\frac{1}{2}\left(1 - \frac{P_{Tx}}{P_Y}\right)}$$

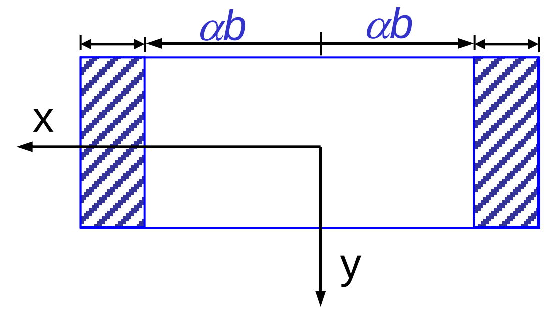
$$\therefore \lambda_x^2 = \sqrt{2\left(1 - \frac{P_{Tx}}{P_Y}\right)} \Bigg/ \frac{P_{Tx}}{P_Y}$$



If inelastic buckling occurs about y – axis

$$P_{cr} = P_{Ty} = \frac{\pi^2 E}{L^2}(2\alpha b)^3 \frac{d}{12}$$

$$\therefore P_{Ty} = \frac{\pi^2 E I_y}{L^2}(2\alpha)^3$$



$$\therefore P_{Ty} = P_y \times \left[2\sqrt{\frac{1}{2}\left(1 - \frac{P_{cr}}{P_Y}\right)}\right]^3$$

$$\therefore P_{Ty} = P_y \times \left[\sqrt{2\left(1 - \frac{P_{Ty}}{P_Y}\right)}\right]^3 \qquad \because P_{cr} = P_{Ty}$$

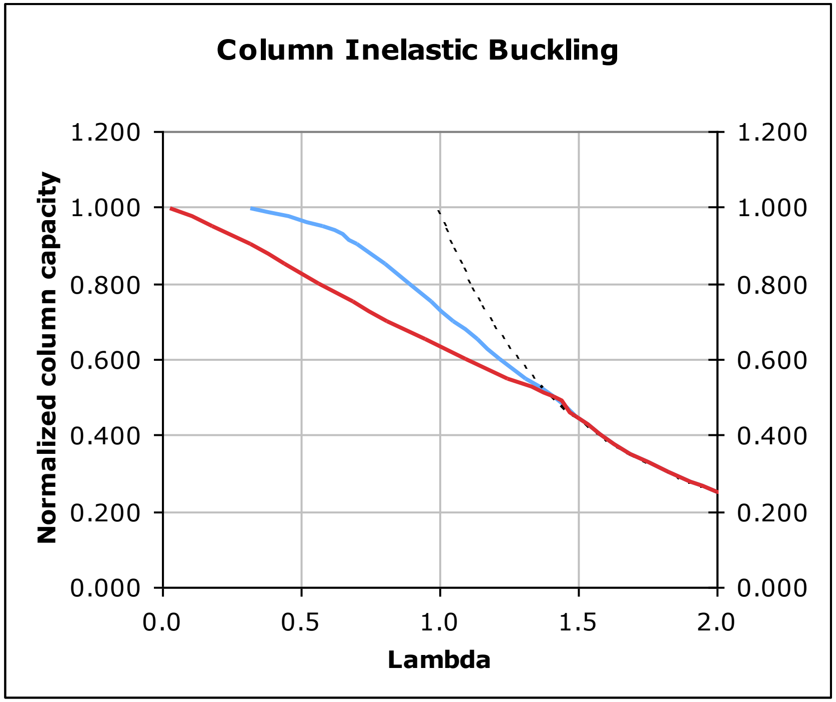
$$\therefore \frac{P_{Ty}}{P_Y} = \frac{P_y}{P_Y} \times \left[\sqrt{2\left(1 - \frac{P_{Ty}}{P_Y}\right)}\right]^3 \qquad Let, \frac{P_y}{P_Y} = \frac{1}{\lambda_y^2} = \pi^2 \frac{E}{\sigma_Y}\left(\frac{r_y}{K_y L_y}\right)^2$$

$$\therefore \frac{P_{Ty}}{P_Y} = \frac{1}{\lambda_y^2} \times \left[\sqrt{2\left(1 - \frac{P_{Ty}}{P_Y}\right)}\right]^3$$

$$\therefore \lambda_y^2 = \left[\sqrt{2\left(1 - \frac{P_{Ty}}{P_Y}\right)}\right]^3 \Bigg/ \frac{P_{Ty}}{P_Y}$$

Residual Stress Effects

P/P_Y	λ_x	λ_y
0.200	2.236	2.236
0.250	2.000	2.000
0.300	1.826	1.826
0.350	1.690	1.690
0.400	1.581	1.581
0.450	1.491	1.491
0.500	1.414	1.414
0.550	1.313	1.246
0.600	1.221	1.092
0.650	1.135	0.949
0.700	1.052	0.815
0.750	0.971	0.687
0.800	0.889	0.562
0.850	0.803	0.440
0.900	0.705	0.315
0.950	0.577	0.182
0.995	0.317	0.032



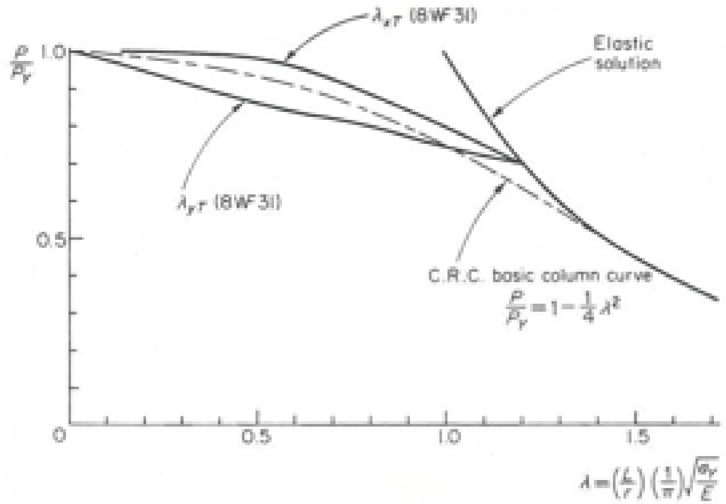
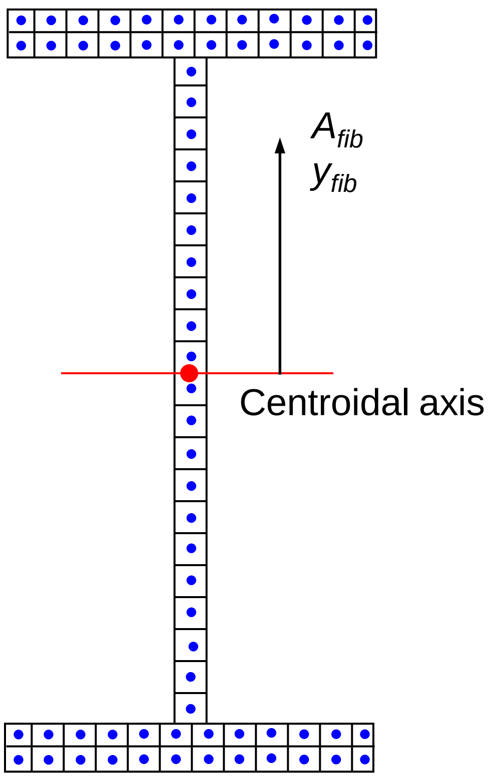


Fig. 4.34. Tangent modulus buckling curves for strong and weak axis buckling of wide-flange columns

Tangent modulus buckling - Numerical



1. *Discretize the cross-section into fibers. Think about the discretization. Do you need the flange to be discretized along the length and width?*
2. *For each fiber, save the area of fiber (A_{fib}), the distances from the centroid y_{fib} and x_{fib}, $I_{x\text{-}fib}$ and $I_{y\text{-}fib}$, the fiber number in the matrix.*
3. *Discretize residual stress distribution*
4. *Calculate residual stress ($\sigma_{r\text{-}fib}$) each fiber*
5. *Check that sum($\sigma_{r\text{-}fib} A_{fib}$) for Section = zero*

Tangent Modulus Buckling - Numerical

6 Calculate effective residual strain (ε_r) for each fiber
$\varepsilon_r = \sigma_r / E$

7 Assume centroidal strain
ε

8 Calculate total strain for each fiber
$\varepsilon_{tot} = \varepsilon + \varepsilon_r$

9 Assume a material stress-strain curve for each fiber

10 Calculate stress in each fiber σ_{fib}

11 Calculate Axial Force = P
Sum $(\sigma_{fib} A_{fib})$

12 Calculate average stress = $\sigma = P/A$

13 Calculate the tangent $(EI)_{TX}$ and $(EI)_{TY}$ for the σ
$(EI)_{TX} = sum(E_{T\text{-}fib}\{y_{fib}^2 A_{fib} + I_{x\text{-}fib}\})$
$(EI)_{Ty} = sum(E_{T\text{-}fib}\{x_{fib}^2 A_{fib} + I_{y\text{-}fib}\})$

14 Calculate the critical $(KL)_X$ and $(KL)_Y$ for the σ
$(KL)_{X\text{-}cr} = \pi \, sqrt\,[(EI)_{TX}/P]$
$(KL)_{y\text{-}cr} = \pi \, sqrt\,[(EI)_{Ty}/P]$

Tangent modulus buckling - numerical

Section Dimension

b	12
d	4
σ_y	50
No. of fibers	20

A	48
Ix	64
Iy	576.00

fiber no.	A_{fib}	x_{fib}	y_{fib}	$\sigma_{r\text{-}fib}$	$\varepsilon_{r\text{-}fib}$	Ix_{fib}	Iy_{fib}
1	2.4	-5.7	0	-22.5	-7.759E-04	3.2	78.05
2	2.4	-5.1	0	-17.5	-6.034E-04	3.2	62.50
3	2.4	-4.5	0	-12.5	-4.310E-04	3.2	48.67
4	2.4	-3.9	0	-7.5	-2.586E-04	3.2	36.58
5	2.4	-3.3	0	-2.5	-8.621E-05	3.2	26.21
6	2.4	-2.7	0	2.5	8.621E-05	3.2	17.57
7	2.4	-2.1	0	7.5	2.586E-04	3.2	10.66
8	2.4	-1.5	0	12.5	4.310E-04	3.2	5.47
9	2.4	-0.9	0	17.5	6.034E-04	3.2	2.02
10	2.4	-0.3	0	22.5	7.759E-04	3.2	0.29
11	2.4	0.3	0	22.5	7.759E-04	3.2	0.29
12	2.4	0.9	0	17.5	6.034E-04	3.2	2.02
13	2.4	1.5	0	12.5	4.310E-04	3.2	5.47
14	2.4	2.1	0	7.5	2.586E-04	3.2	10.66
15	2.4	2.7	0	2.5	8.621E-05	3.2	17.57
16	2.4	3.3	0	-2.5	-8.621E-05	3.2	26.21
17	2.4	3.9	0	-7.5	-2.586E-04	3.2	36.58
18	2.4	4.5	0	-12.5	-4.310E-04	3.2	48.67
19	2.4	5.1	0	-17.5	-6.034E-04	3.2	62.50
20	2.4	5.7	0	-22.5	-7.759E-04	3.2	78.05

Tangent Modulus Buckling - numerical

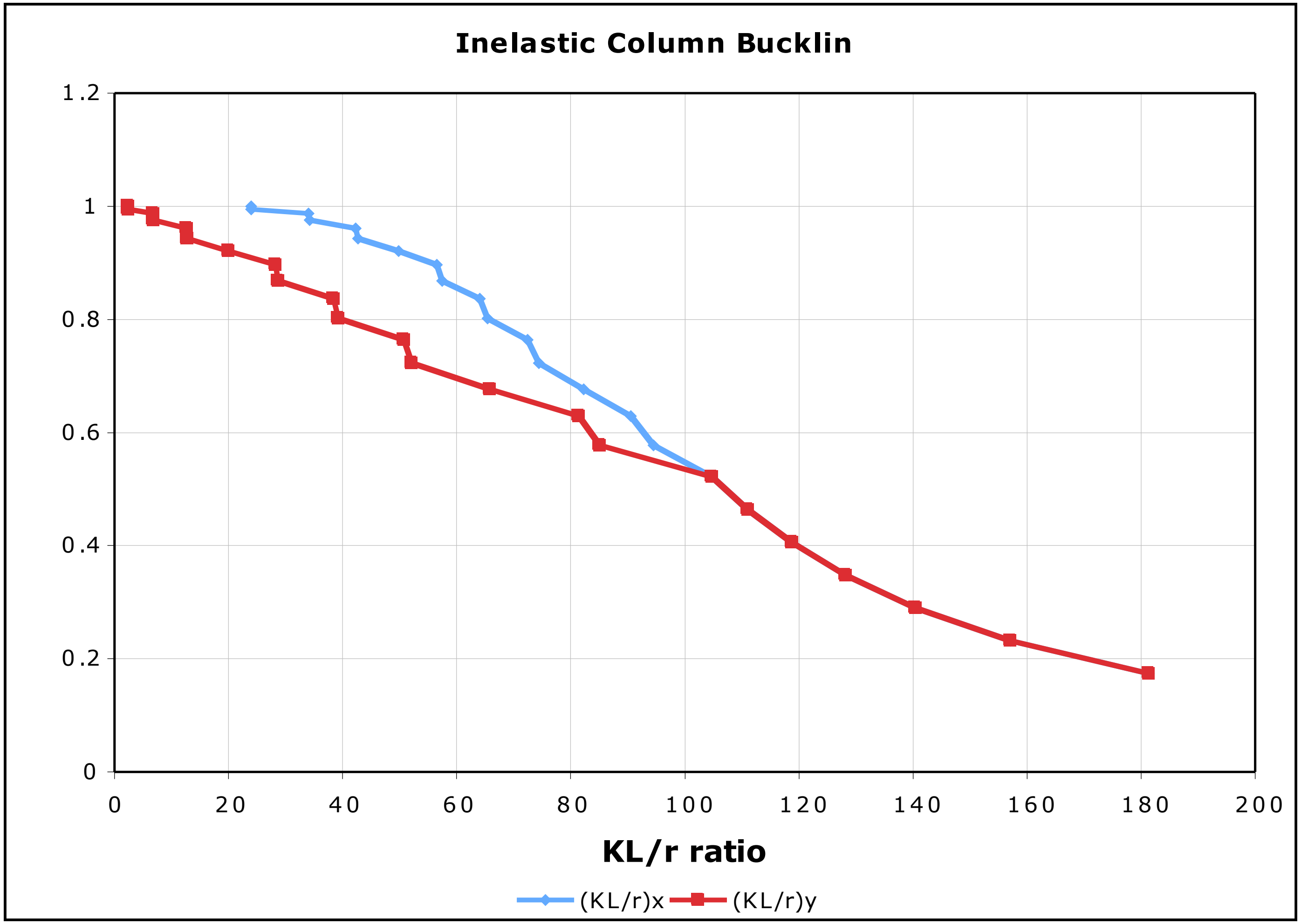
Strain Increment

$\Delta\varepsilon$	Fiber no.	ε_{tot}	σ_{fib}	E_{fib}	$EI_{Tx\text{-}fib}$	$EI_{Ty\text{-}fib}$	P_{fib}
-0.0003	1	-1.076E-03	-31.2	29000	92800	2.26E+06	-74.88
	2	-9.034E-04	-26.2	29000	92800	1.81E+06	-62.88
	3	-7.310E-04	-21.2	29000	92800	1.41E+06	-50.88
	4	-5.586E-04	-16.2	29000	92800	1.06E+06	-38.88
	5	-3.862E-04	-11.2	29000	92800	7.60E+05	-26.88
	6	-2.138E-04	-6.2	29000	92800	5.09E+05	-14.88
	7	-4.138E-05	-1.2	29000	92800	3.09E+05	-2.88
	8	1.310E-04	3.8	29000	92800	1.59E+05	9.12
	9	3.034E-04	8.8	29000	92800	5.85E+04	21.12
	10	4.759E-04	13.8	29000	92800	8.35E+03	33.12
	11	4.759E-04	13.8	29000	92800	8.35E+03	33.12
	12	3.034E-04	8.8	29000	92800	5.85E+04	21.12
	13	1.310E-04	3.8	29000	92800	1.59E+05	9.12
	14	-4.138E-05	-1.2	29000	92800	3.09E+05	-2.88
	15	-2.138E-04	-6.2	29000	92800	5.09E+05	-14.88
	16	-3.862E-04	-11.2	29000	92800	7.60E+05	-26.88
	17	-5.586E-04	-16.2	29000	92800	1.06E+06	-38.88
	18	-7.310E-04	-21.2	29000	92800	1.41E+06	-50.88
	19	-9.034E-04	-26.2	29000	92800	1.81E+06	-62.88
	20	-1.076E-03	-31.2	29000	92800	2.26E+06	-74.88

Tangent Modulus Buckling - Numerical

$\Delta\varepsilon$	P	EI_{Tx}	EI_{Ty}	KL_{x-cr}	KL_{y-cr}	σ_T/σ_Y	$(KL/r)_x$	$(KL/r)_y$
–0.0003	-417.6	1856000	16704000	209.4395102	628.3185307	0.174	181.3799364	181.3799364
–0.0004	-556.8	1856000	16704000	181.3799364	544.1398093	0.232	157.0796327	157.0796327
-0.0005	-696	1856000	16704000	162.231147	486.6934411	0.29	140.4962946	140.4962946
-0.0006	-835.2	1856000	16704000	148.0960979	444.2882938	0.348	128.254983	128.254983
-0.0007	-974.4	1856000	16704000	137.1103442	411.3310325	0.406	118.7410412	118.7410412
-0.0008	-1113.6	1856000	16704000	128.254983	384.764949	0.464	111.0720735	111.0720735
-0.0009	-1252.8	1856000	16704000	120.9199576	362.7598728	0.522	104.7197551	104.7197551
-0.001	-1384.8	1670400	12177216	109.11051	294.5983771	0.577	94.49247352	85.04322617
-0.0011	-1510.08	1670400	12177216	104.4864889	282.1135199	0.6292	90.48795371	81.43915834
-0.0012	-1624.32	1484800	8552448	94.98347542	227.960341	0.6768	82.25810265	65.80648212
-0.0013	-1734.72	1299200	5729472	85.97519823	180.5479163	0.7228	74.45670576	52.11969403
-0.0014	-1832.16	1299200	5729472	83.65775001	175.681275	0.7634	72.44973673	50.71481571
-0.0015	-1924.8	1113600	3608064	75.56517263	136.0173107	0.802	65.44135914	39.26481548
-0.0016	-2008.32	1113600	3608064	73.97722346	133.1590022	0.8368	64.06615482	38.43969289
-0.0017	-2083.2	928000	2088000	66.30684706	99.46027059	0.868	57.423414	28.711707
-0.0018	-2152.8	928000	2088000	65.22619108	97.83928663	0.897	56.48753847	28.24376924
-0.0019	-2209.92	742400	1069056	57.58118233	69.0974188	0.9208	49.86676668	19.94670667
-0.002	-2263.2	556800	451008	49.27629185	44.34866267	0.943	42.67452055	12.80235616
-0.0021	-2304.96	556800	451008	48.8278711	43.94508399	0.9604	42.28617679	12.68585304
-0.0022	-2340.48	371200	133632	39.56410897	23.73846538	0.9752	34.26352344	6.852704688
-0.0023	-2368.32	371200	133632	39.33088015	23.59852809	0.9868	34.06154136	6.812308273
-0.0024	-2386.08	185600	16704	27.70743725	8.312231176	0.9942	23.99534453	2.399534453
-0.00249	-2398.608	185600	16704	27.63498414	8.290495243	0.99942	23.9325983	2.39325983

Tangent Modulus Buckling - Numerical



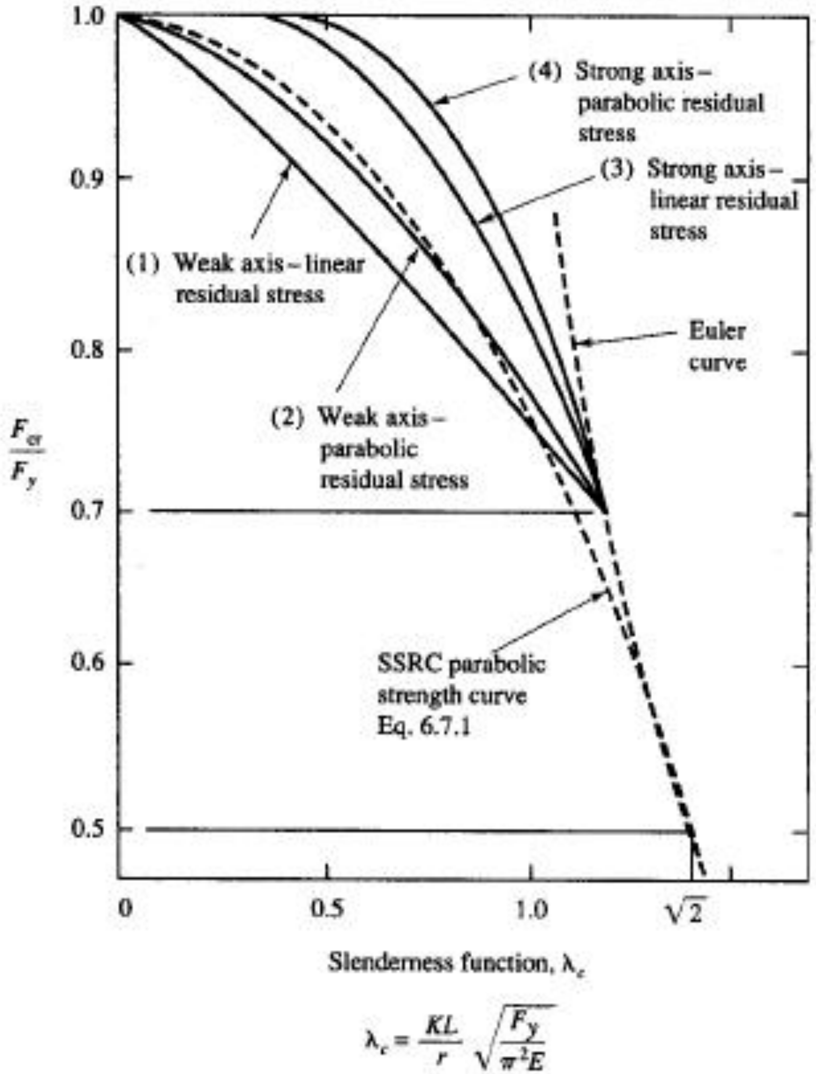


Figure 6.7.1
Column strength curves for H-shaped sections having compressive residual stress at flange tips. (Adapted from Ref. 6.20, p. 39)

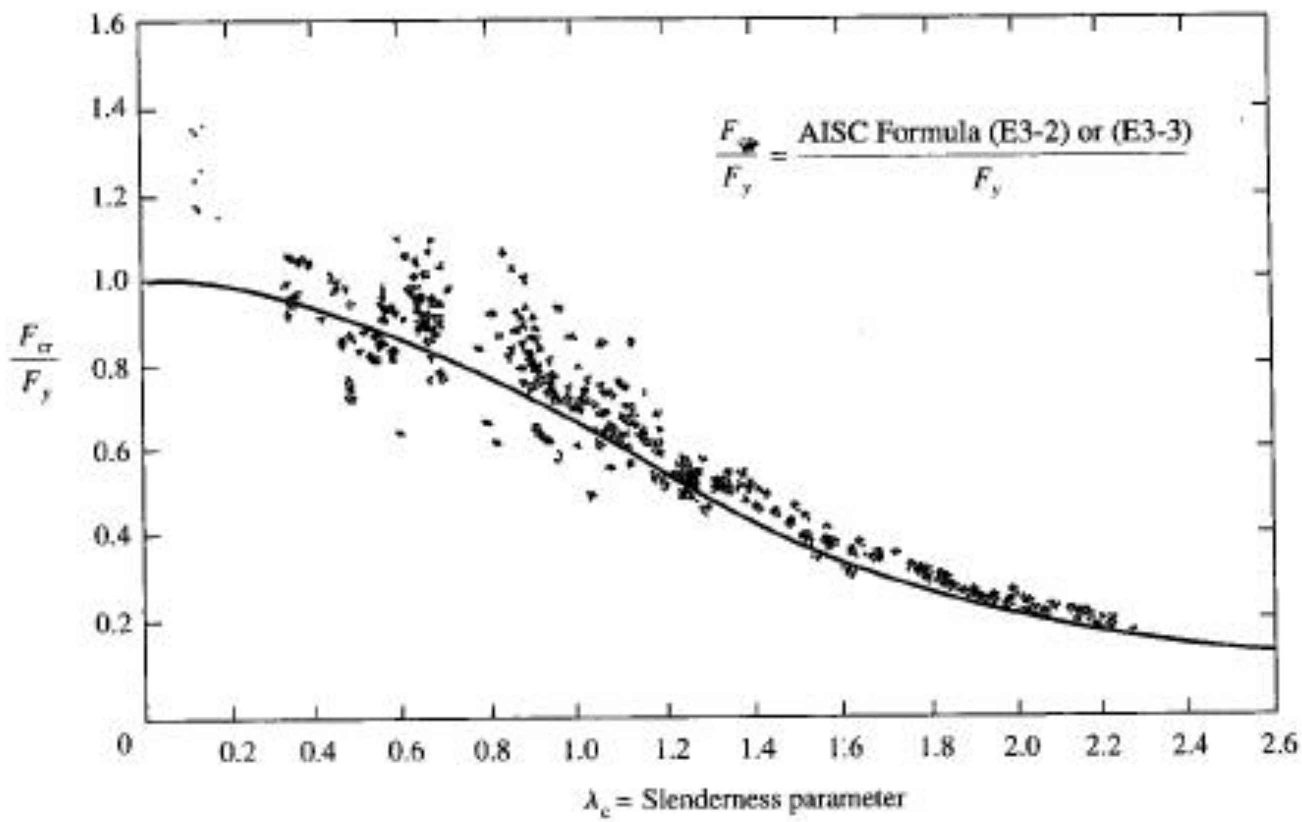
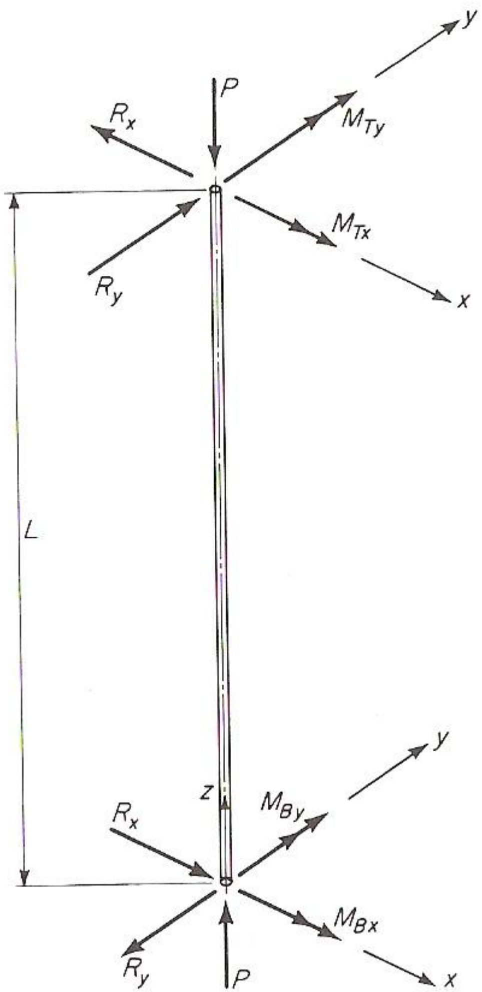


Figure 6.7.2
Comparison of AISC equations for F_{cr} for columns with data from physical tests. (Test data from Hall [6.24])

ELASTIC BUCKLING OF BEAMS

- Going back to the original three second-order differential equations:



Therefore,

1 $$E I_x v'' + P v - \phi\left(P x_0 + M_{BY} - \frac{z}{L}(M_{TY} + M_{BY})\right) = M_{BX} - \frac{z}{L}(M_{TX} + M_{BX})$$

2 $$E I_y u'' + P u - \phi\left(-P y_0 + M_{BX} - \frac{z}{L}(M_{TX} + M_{BX})\right) = -M_{BY} + \frac{z}{L}(M_{TY} + M_{BY})$$

3 $$E I_w \phi''' - (G K_T + \bar{K})\phi' + u'\left(-M_{BX} - \frac{z}{L}(M_{BX} + M_{TX}) + P y_0\right)$$

$$- v'\left(M_{BY} + \frac{z}{L}(M_{BY} + M_{TY}) + P x_0\right) - \frac{v}{L}(M_{TY} + M_{BY}) - \frac{u}{L}(M_{TX} + M_{BX}) = 0$$

ELASTIC BUCKLING OF BEAMS

- Consider the case of a beam subjected to uniaxial bending only:
  - because most steel structures have beams in uniaxial bending
  - Beams under biaxial bending do not undergo elastic buckling
- P=0; $M_{TY}=M_{BY}=0$
- The three equations simplify to:

1

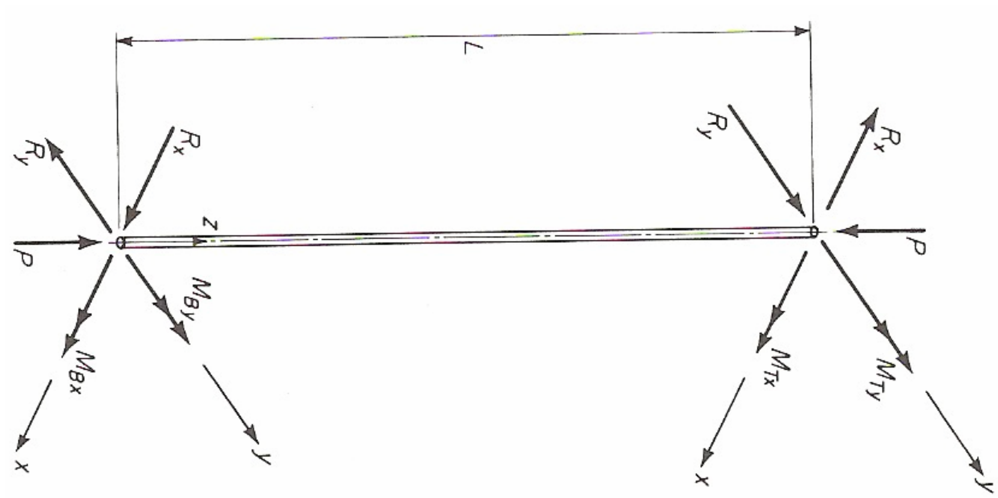
2    $(-\phi)$

3

- Equation (1) is an uncoupled differential equation describing in-plane bending behavior caused by M_{TX} and M_{BX}

ELASTIC BUCKLING OF BEAMS

- Equations (2) and (3) are coupled equations in u and ϕ – that describe the lateral bending and torsional behavior of the beam. In fact they define the lateral torsional buckling of the beam.
- The beam must satisfy all three equations (1, 2, and 3). Hence, beam in-plane bending will occur UNTIL the lateral torsional buckling moment is reached, when it will take over.
- Consider the case of uniform moment (M_o) causing compression in the top flange. This will mean that
  - $-M_{BX} = M_{TX} = M_o$



ELASTIC BUCKLING OF BEAMS

- For this case, the differential equations (2 and 3) will become:

$$E\,I_y\,u'' + \phi\,M_o = 0$$

$$E\,I_w\,\phi''' - (G\,K_T + \bar{K})\,\phi' + u'\left(M_o\right) = 0$$

$$where:$$

$$\bar{K} = Wagner's\ effect\ due\ to\ warping\ caused\ by\ torsion$$

$$\bar{K} = \int_A \sigma\,a^2\,dA$$

$$But, \sigma = \frac{M_o}{I_x}\,y \quad \Rightarrow neglecting\ higher\ order\ terms$$

$$\therefore \bar{K} = \int_A \frac{M_o}{I_x}\,y\left[(x_o - x)^2 + (y_o - y)^2\right]dA$$

$$\therefore \bar{K} = \frac{M_o}{I_x}\int_A y\left[x_o^2 + x^2 - 2xx_0 + y_o^2 + y^2 - 2yy_0\right]dA$$

$$\therefore \bar{K} = \frac{M_o}{I_x}\left[x_o^2\int_A y\ \ dA + \int_A y\left[x^2 + y^2\right]dA - x_0\int_A 2xy\ dA + y_o^2\int_A y\ \ dA - 2y_o\int_A y^2 dA\right]$$

ELASTIC BUCKLING OF BEAMS

$$\therefore \bar{K} = \frac{M_o}{I_x}\left[\int_A y\left[x^2+y^2\right]dA - 2y_o I_x\right]$$

$$\therefore \bar{K} = M_o\left[\frac{\int_A y\left[x^2+y^2\right]dA}{I_x} - 2y_o\right]$$

$$\therefore \bar{K} = M_o\beta_x \qquad \Rightarrow where,\ \beta_x = \frac{\int_A y\left[x^2+y^2\right]dA}{I_x} - 2y_o$$

β_x *is a new sectional property*

The beam buckling differential equations become:

$$(2) \quad E\, I_y\, u'' + \phi\, M_o = 0$$

$$(3) \quad E\, I_w\, \phi''' - (G\, K_T + M_o\beta_x)\,\phi' + u'\left(M_o\right) = 0$$

ELASTIC BUCKLING OF BEAMS

$$Equation\ (2)\ gives\ \ u'' = -\frac{M_o}{E\ I_y}\phi$$

$$Substituting\ u''\ \ from\ Equation\ (2)\ in\ (3)\ gives:$$

$$E\ I_w\ \phi^{iv} - (G\ K_T + M_o \beta_x)\ \phi'' - \frac{M_o^{\ 2}}{E\ I_y}\phi = 0$$

$$For\ doubly\ symmetric\ section: \beta_x = 0$$

$$\therefore \phi^{iv} - \frac{G\ K_T}{E\ I_w}\phi'' - \frac{M_o^{\ 2}}{E^2\ I_y I_w}\phi = 0$$

$$Let,\ \lambda_1 = \frac{G\ K_T}{E\ I_w} \quad and \quad \lambda_2 = \frac{M_o^{\ 2}}{E^2\ I_y I_w}$$

$$\therefore \phi^{iv} - \lambda_1\ \phi'' - \lambda_2 \phi = 0 \quad \Rightarrow becomes\ the\ combined\ d.e.\ of\ \ LTB$$

ELASTIC BUCKLING OF BEAMS

$$Assume\ solution\ is\ of\ the\ form\quad \phi = e^{\lambda z}$$

$$\therefore \left(\lambda^4 - \lambda_1\,\lambda^2 - \lambda_2\right)e^{\lambda z} = 0$$

$$\therefore \lambda^4 - \lambda_1\,\lambda^2 - \lambda_2 = 0$$

$$\therefore \lambda^2 = \frac{\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2}\quad ,\quad -\frac{\sqrt{\lambda_1^2 + 4\lambda_2} - \lambda_1}{2}$$

$$\therefore \lambda = \pm\sqrt{\frac{\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2}}\quad ,\quad \pm i\sqrt{\frac{\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2}}$$

$$\therefore Let,\ \lambda = \pm\alpha_1\quad ,\quad and\quad \pm i\,\alpha_2$$

Above are the four roots for λ

$$\therefore \phi = C_1 e^{\alpha_1 z} + C_2 e^{-\alpha_1 z} + C_3 e^{i\alpha_2 z} + C_4 e^{-i\alpha_2 z}$$

$\therefore$ *collecting real and imaginary terms*

$$\therefore \phi = G_1\cosh(\alpha_1 z) + G_2\sinh(\alpha_1 z) + G_3\sin(\alpha_2 z) + G_4\cos(\alpha_2 z)$$

ELASTIC BUCKLING OF BEAMS

- Assume simply supported boundary conditions for the beam:

$$\therefore \phi(0) = \phi''(0) = \phi(L) = \phi''(L) = 0$$

Solution for ϕ *must satisfy all four b.c.*

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ \alpha_1^2 & 0 & 0 & -\alpha_2^2 \\ \cosh(\alpha_1 L) & \sinh(\alpha_1 L) & \sin(\alpha_2 L) & \cos(\alpha_2 L) \\ \alpha_1^2 \cosh(\alpha_1 L) & \alpha_1^2 \sinh(\alpha_1 L) & -\alpha_2^2 \sin(\alpha_2 L) & -\alpha_2^2 \cos(\alpha_2 L) \end{bmatrix} \times \begin{Bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{Bmatrix} = 0$$

For buckling coefficient matrix must be singular:

$$\therefore \text{determinant of matrix} = 0$$

$$\therefore \left(\alpha_1^2 + \alpha_2^2\right) \times \sinh(\alpha_1 L) \times \sin(\alpha_2 L) = 0$$

Of these:

$$only \quad \sin(\alpha_2 L) = 0$$

$$\therefore \alpha_2 L = n\pi$$

ELASTIC BUCKLING OF BEAMS

$$\therefore \alpha_2 = \frac{n\pi}{L}$$

$$\therefore \sqrt{\frac{\sqrt{\lambda_1^2 + 4\lambda_2} - \lambda_1}{2}} = \frac{\pi}{L}$$

$$\therefore \sqrt{\lambda_1^2 + 4\lambda_2} - \lambda_1 = \frac{2\pi^2}{L^2}$$

$$\therefore \lambda_2 = \frac{\left(\frac{2\pi^2}{L^2} + \lambda_1\right)^2 - \lambda_1^2}{4} = \frac{\left(\frac{2\pi^2}{L^2} + 2\lambda_1\right)\left(\frac{2\pi^2}{L^2}\right)}{4}$$

$$\therefore \lambda_2 = \left(\frac{\pi^2}{L^2} + \lambda_1\right)\left(\frac{\pi^2}{L^2}\right)$$

$$\therefore \lambda_2 = \frac{M_o^{\;2}}{E^2\; I_y I_w} = \left(\frac{\pi^2}{L^2} + \frac{G\,K_T}{E\,I_w}\right)\left(\frac{\pi^2}{L^2}\right)$$

$$\therefore M_o = \sqrt{\left(E^2\; I_y I_w\right)\left(\frac{\pi^2}{L^2} + \frac{G\,K_T}{E\,I_w}\right)\left(\frac{\pi^2}{L^2}\right)}$$

$$\therefore M_o = \sqrt{\frac{\pi^2 E\,I_y}{L^2}\left(\frac{\pi^2 E\,I_w}{L^2} + G\,K_T\right)}$$